

DIFFERENCE HIERARCHIES FOR $\widehat{GL}_\infty^{(n)}$ TAU-FUNCTIONS

DARLAYNE ADDABBO AND MAARTEN BERGVELT

ABSTRACT. We introduce hierarchies of difference equations (referred to as nT -systems) associated to the action of a (centrally extended) infinite matrix group $\widehat{GL}_\infty^{(n)}$ on n -component fermionic Fock space. The solutions are given by matrix elements (τ -functions) for this action. Here, we mainly discuss the $2T$ - and $3T$ -systems, and give conjectures for the general nT -systems. It seems that the nT -systems are related to the fusion rules for quantum transfer matrices parametrized by Young diagrams with $n - 1$ rectangular blocks. ([Zab97],[LWZ97]).

The $2T$ -system is, after a change of variables, the usual T -system of type A . Restriction from $\widehat{GL}_\infty^{(n)}$ to the subgroup LGL_n defines nQ -systems, studied earlier by the present authors for $n = 2, 3$.

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1. INTRODUCTION

Recently there has been much interest in discrete integrable systems, difference equations with many conserved quantities, in the context of cluster algebras. (See for instance [Ked08], [DFK09].) In particular the T -system and its reduction the Q -system has been studied in great detail, with applications in representation theory and combinatorics. It is known that the T - and Q -systems are discrete Hirota equations.

Now experience in the general theory of integrable systems has shown that Hirota equations don't occur in isolation: they are found to appear in families, forming hierarchies of compatible equations. Also experience shows that such integrable hierarchies are attached to representation theoretic data. For instance, the KP hierarchy corresponds to the principal construction of the basic representation of the infinite matrix group GL_∞ . The KdV hierarchy, a reduction of the KP hierarchy, is similarly related to LSL_2 , the loop group of sl_2 . LSL_2 is a subgroup of GL_∞ .

It is then a natural question to ask whether the T -system and the Q -system are part of more general hierarchies of difference equations, and whether one can give a representation theoretic construction of these hierarchies. The aim of this paper is to show that the answer to both questions is positive. There are both T and Q hierarchies of which the T and Q systems are the simplest members. Furthermore, these hierarchies are also attached to GL_∞ and LSL_2 in the same way as for the KP and KdV case: we propose the slogan: " T is to Q as KP is to KdV".

The difference between the KP hierarchy and the T hierarchy (both connected to GL_∞) lies in the choice of bosonization. It is well known that different KP like hierarchies can be constructed depending on the choice of a Heisenberg subalgebra in the (centrally extended) infinite matrix algebra, each inequivalent Heisenberg algebra giving rise to a different bosonization. In this paper we do something slightly different, we choose a nilpotent subgroup, instead of a Heisenberg subalgebra.

Let us briefly sketch the construction of the KP hierarchy in order to compare it to the construction in the present paper. For the KP hierarchy, one can start with the fermionic Fock space F , the semi-infinite wedge space. On F we have the actions of the matrix algebra gl_∞ and group GL_∞ , and the centrally extended completion A_∞ of gl_∞ . In A_∞ there is a Heisenberg subalgebra spanned by elements $\alpha(k)$, such that

$$[\alpha(k), \alpha(l)] = k\delta_{k,l}.$$

Also, on F we have a bilinear form \langle, \rangle such that elementary wedges are orthonormal. Finally, fermionic Fock space has a grading by charge, $F = \bigoplus_{k \in \mathbb{Z}} F_k$, where F_k is the charge k subspace. Then F_0 contains the vacuum vector v_0 . Then one defines a map, called bosonization,

$$\Phi: F_0 \rightarrow \mathbb{C}[t_1, t_2, \dots], \quad \omega \mapsto \langle v_0, e^{\sum_{k=1}^{\infty} \alpha(k) \omega} \rangle.$$

In case $\omega \in F_0$ is an element of the GL_∞ orbit of the vacuum, $\omega = gv_0$, we call $\Phi(\omega)$ the τ -function $\tau_\omega(t)$, where $t = (t_1, t_2, \dots)$ are the KP times. One of the points of the construction is that $\tau_\omega(t)$ solves an infinite collection of differential equations, the KP hierarchy.

This construction is an instance of the general philosophy of special functions: interesting functions are matrix elements of an action of a group¹ G on some representation V , with the variables in the special function obtained by choosing coordinates on G (or a subgroup,

¹Or some other algebraic structure such as a quantum group, etc.

say diagonal elements in a matrix group). One can think of the KP times $t = (t_1, t_2, \dots)$ as coordinates on a subgroup of GL_∞ (of matrices that are constant along diagonals).

Now the observation is that for the T -system one can choose a different subgroup of GL_∞ , consisting of certain lower triangular matrices, with coordinates $c_{k,l}$, $k, l \in \mathbb{Z}$, $k, l \geq 0$. By more or less the same construction as for the KP hierarchy, one defines tau-functions and shows that the tau-functions satisfy difference equations, one of which is the original T -system equation.

In our earlier work, we showed that the $2Q$ -system yielded the usual type A Q -system [AB]: For all $k \geq 0$, $\alpha \in \mathbb{Z}$,

$$(2Q\text{-system}) \quad (\tau_k^{(\alpha)})^2 = \tau_k^{(\alpha+1)} \tau_k^{(\alpha-1)} - \tau_{k+1}^{(\alpha-1)} \tau_{k-1}^{(\alpha+1)}.$$

The $2T$ -system (described in this paper), yields the type A T -system: For all $k \geq 0$ and $\alpha, \beta \in \mathbb{Z}$,

$$(2T\text{-system}) \quad \tau_k^{(\alpha+1, \beta)} \tau_k^{(\alpha, \beta+1)} = \tau_{k+1}^{(\alpha, \beta+1)} \tau_{k-1}^{(\alpha+1, \beta)} + \tau_k^{(\alpha, \beta)} \tau_k^{(\alpha+1, \beta+1)}.$$

In our previous work, we showed that the $3Q$ -system is: For all $k, \ell \geq 0$, $\alpha, \beta \in \mathbb{Z}$,

$$(3Q\text{-system}) \quad \begin{aligned} \tau_{k-1, \ell-1}^{(\alpha+1, \beta)} \tau_{k+1, \ell}^{(\alpha, \beta)} + \tau_{k, \ell-1}^{(\alpha, \beta)} \tau_{k, \ell}^{(\alpha+1, \beta)} &= \tau_{k, \ell}^{(\alpha, \beta)} \tau_{k, \ell-1}^{(\alpha+1, \beta)} \\ \tau_{k-1, \ell}^{(\alpha, \beta)} \tau_{k, \ell+1}^{(\alpha, \beta+1)} + \tau_{k-1, \ell+1}^{(\alpha, \beta)} \tau_{k, \ell}^{(\alpha, \beta+1)} &= \tau_{k-1, \ell}^{(\alpha, \beta+1)} \tau_{k, \ell+1}^{(\alpha, \beta)} \\ (\tau_{k, \ell}^{(\alpha+1, \beta)})^2 &= \tau_{k, \ell}^{(\alpha, \beta)} \tau_{k, \ell}^{(\alpha+2, \beta)} + \tau_{k+1, \ell+1}^{(\alpha, \beta)} \tau_{k-1, \ell-1}^{(\alpha+2, \beta)} - \tau_{k+1, \ell}^{(\alpha, \beta)} \tau_{k-1, \ell}^{(\alpha+2, \beta)} \\ (\tau_{k, \ell}^{(\alpha, \beta+1)})^2 &= \tau_{k, \ell}^{(\alpha, \beta)} \tau_{k, \ell}^{(\alpha, \beta+2)} - \tau_{k, \ell-1}^{(\alpha, \beta+2)} \tau_{k, \ell+1}^{(\alpha, \beta)} - \tau_{k+1, \ell}^{(\alpha, \beta+2)} \tau_{k-1, \ell}^{(\alpha, \beta)}. \end{aligned}$$

In the following, we prove that the $3T$ -system includes the equations, for all $k, \ell \geq 0$ and $\alpha, \beta, \gamma \in \mathbb{Z}$,

$$(3T\text{-system}) \quad \begin{aligned} \tau_{k, \ell}^{(\alpha+1, \beta, \gamma)} \tau_{k, \ell+1}^{(\alpha, \beta, \gamma)} &= \tau_{k, \ell}^{(\alpha, \beta, \gamma)} \tau_{k, \ell+1}^{(\alpha+1, \beta, \gamma)} + \tau_{k+1, \ell+1}^{(\alpha, \beta, \gamma)} \tau_{k-1, \ell}^{(\alpha+1, \beta, \gamma)} \\ \tau_{k, \ell}^{(\alpha, \beta+1, \gamma)} \tau_{k+1, \ell+1}^{(\alpha, \beta, \gamma)} &= \tau_{k, \ell}^{(\alpha, \beta, \gamma)} \tau_{k+1, \ell+1}^{(\alpha, \beta+1, \gamma)} + \tau_{k, \ell+1}^{(\alpha, \beta, \gamma)} \tau_{k+1, \ell}^{(\alpha, \beta+1, \gamma)} \\ \tau_{k, \ell}^{(\alpha, \beta, \gamma+1)} \tau_{k+1, \ell}^{(\alpha, \beta, \gamma)} &= \tau_{k, \ell}^{(\alpha, \beta, \gamma)} \tau_{k+1, \ell}^{(\alpha, \beta, \gamma+1)} + \tau_{k+1, \ell+1}^{(\alpha, \beta, \gamma+1)} \tau_{k, \ell-1}^{(\alpha, \beta, \gamma)} \\ \tau_{k, \ell}^{(\alpha, \beta, \gamma)} \tau_{k, \ell}^{(\alpha+1, \beta+1, \gamma+1)} &= \tau_{k, \ell}^{(\alpha+1, \beta, \gamma)} \tau_{k, \ell}^{(\alpha, \beta+1, \gamma+1)} + \tau_{k+1, \ell+1}^{(\alpha, \beta+1, \gamma+1)} \tau_{k-1, \ell-1}^{(\alpha+1, \beta, \gamma)} - \tau_{k+1, \ell}^{(\alpha, \beta+1, \gamma+1)} \tau_{k-1, \ell}^{(\alpha+1, \beta, \gamma)} \\ \tau_{k, \ell}^{(\alpha, \beta, \gamma)} \tau_{k, \ell}^{(\alpha+1, \beta+1, \gamma+1)} &= \tau_{k, \ell}^{(\alpha, \beta+1, \gamma)} \tau_{k, \ell}^{(\alpha+1, \beta, \gamma+1)} - \tau_{k, \ell+1}^{(\alpha+1, \beta, \gamma+1)} \tau_{k, \ell-1}^{(\alpha, \beta+1, \gamma)} - \tau_{k-1, \ell}^{(\alpha+1, \beta, \gamma+1)} \tau_{k+1, \ell}^{(\alpha, \beta+1, \gamma)} \\ \tau_{k, \ell}^{(\alpha, \beta, \gamma)} \tau_{k, \ell}^{(\alpha+1, \beta+1, \gamma+1)} &= \tau_{k, \ell}^{(\alpha, \beta, \gamma+1)} \tau_{k, \ell}^{(\alpha+1, \beta+1, \gamma)} + \tau_{k-1, \ell-1}^{(\alpha+1, \beta+1, \gamma)} \tau_{k+1, \ell+1}^{(\alpha, \beta, \gamma+1)} - \tau_{k, \ell-1}^{(\alpha+1, \beta+1, \gamma)} \tau_{k, \ell+1}^{(\alpha, \beta, \gamma+1)}. \end{aligned}$$

In general, we have systems of difference equations for each \widehat{GL}_n and each $\widehat{GL}_\infty^{(n)}$. We will refer to our \widehat{GL}_n and $\widehat{GL}_\infty^{(n)}$ systems as nQ - and nT -systems, respectively. While this paper focuses primarily on $n = 2, 3$, we will prove that the nQ - and nT - τ -functions satisfy bilinear equations with only three terms. We will also present conjectures for length $n + 1$ bilinear equations satisfied by the nQ - and nT τ -functions, which we hope to prove in future work.

2. n COMPONENT SEMI INFINITE WEDGE SPACE

Most of the content of the next two sections can be found in our previous paper [AB], see also [tKvdL91], but is reproduced here for the reader's convenience.

Let $e_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$, $e_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}$, \dots , $e_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ be the standard basis of \mathbb{C}^n and let $H^{(n)} =$

$\mathbb{C}^n \otimes \mathbb{C}[z, z^{-1}]$, with basis $e_a^k = e_a z^k$, for $a = 0, 1, \dots, n-1$ and $k \in \mathbb{Z}$.

Let $F^{(n)}$ be the fermionic Fock space, the semi infinite wedge space based on $H^{(n)}$. It contains semi-infinite wedges

$$\omega = w_0 \wedge w_1 \wedge w_3 \wedge \dots, \quad w_i \in H^{(n)},$$

where the w_i satisfy some restrictions that we will presently discuss. Semi-infinite wedges obey the usual rules of exterior algebra, like multilinearity in each factor and antisymmetry under exchange of two factors.

To formulate the restrictions on the w_i appearing in the wedge ω above, we introduce the Clifford algebra $Cl^{(n)}$ acting on $F^{(n)}$: it is generated by wedging operators $e_a z^k \wedge$, $0 \leq a \leq n-1$, $k \in \mathbb{Z}$ and their adjoints, the contracting operators $i(e_a z^k)$ (defined by $i(e_a z^k)\alpha = \beta$ if $e_a z^k \wedge \beta = \alpha$.)

Let v_0 be the vacuum vector

$$v_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \wedge \dots \wedge \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} z \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ z \\ \vdots \\ 0 \end{pmatrix} \wedge \dots$$

Then we define $F^{(n)}$ to be span of the wedges obtained by acting on the vacuum, v_0 , by monomials in the wedging and contracting operators. To get a basis for $F^{(n)}$ we specify an ordering on the wedging/contracting operators acting on $F^{(n)}$. One possibility is the following:

Definition 2.1. An elementary wedge in $F^{(n)}$ is an element $\omega = Mv_0$, where

$$M = M_{n-1} \dots M_1 M_0, \quad M_a = M_a^+ M_a^-, \quad 0 \leq a \leq n-1,$$

where

$$M_a^\pm = {}_a\psi_{(k_1)}^\pm {}_a\psi_{(k_2)}^\pm \dots {}_a\psi_{(k_s)}^\pm, \quad k_1 < k_2 < \dots < k_s \leq 1,$$

is a monomial in ${}_a\psi_{(k)}^\pm$ for $k \leq -1$, ordered in increasing order from left to right. Here, ${}_a\psi_{(k)}^+ = e_a z^k \wedge$ and ${}_a\psi_{(k)}^- = i(e_a z^{-k-1})$.

The statement that the elementary wedges form a basis for $F^{(n)}$ follows from the Poincaré-Birkhoff-Witt theorem for the Lie superalgebra underlying the Clifford algebra.

We define a bilinear form, denoted \langle, \rangle , on $F^{(n)}$ by declaring the elementary wedges to be orthonormal. Then the fermion operators satisfy the following adjointness property

$$(1) \quad \langle {}_a\psi_{(k)}^+ v, w \rangle = \langle e_a z^k \wedge v, w \rangle = \langle v, i(e_a z^k) w \rangle = \langle v, {}_a\psi_{(-k-1)}^- w \rangle,$$

The n -component fermionic Fock space $F^{(n)}$ has a grading by the Abelian group \mathbb{Z}^n , i.e., we have a decomposition $F^{(n)} = \bigoplus_{\delta \in \mathbb{Z}^n} F_\delta^{(n)}$. The vacuum has degree $(0, \dots, 0)$. We introduce a

basis in \mathbb{Z}^n by

$$\delta_a = (0, \dots, 0, 1, 0, \dots, 0), \quad 1 \text{ on the } a\text{th position}, \quad 0 \leq a \leq n-1.$$

The grading on $F^{(n)}$ induces a grading on linear maps on $F^{(n)}$: if $L: F^{(n)} \rightarrow F^{(n)}$ has the property that there exists a $\delta \in \mathbb{Z}^n$ so that for all $\omega \in \mathbb{Z}^n$, L restricts to a map $F_\omega^{(n)} \rightarrow F_{\omega+\delta}^{(n)}$, then we say that L has degree δ . Then the wedging operators $e_a z^k \wedge$ have degree δ_a , and the contracting operators $i(e_a z^k)$ have degree $-\delta_a$. The *total degree* of an element $v \in F^{(n)}$ of degree $(d_0, d_1, \dots, d_{n-1})$ is just the sum $\sum_{a=0}^{n-1} d_a$.

3. FERMION FIELDS

It is useful to collect the generators of the Clifford algebra in generating series. Therefore, define *fermion fields*

$$\psi_a^\pm(w) = \sum_{k \in \mathbb{Z}} {}_a\psi_{(k)}^\pm w^{-k-1}, \quad 0 \leq a \leq n-1.$$

(We follow more or less the convention of [Kac96].)

The fermionic fields, $\psi_a^\pm(z)$ have degree $\pm\delta_a$ and satisfy commutation relations

$$[\psi_a^\pm(z), \psi_b^\pm(w)]_+ = 0, \quad [\psi_a^+(w_1), \psi_b^-(w_2)]_+ = \delta_{ab} \delta(w_1, w_2),$$

where the formal delta distribution is defined by

$$\delta(z, w) = \sum_{k \in \mathbb{Z}} z^k w^{-k-1}.$$

From (1) we find adjointness for fields:

$$(2) \quad \langle \psi_a^+(z)v, w \rangle = \sum_{k \in \mathbb{Z}} \langle {}_a\psi_{(k)}^+ v, w \rangle z^{-k-1} = \sum_{k \in \mathbb{Z}} \langle v, {}_a\psi_{(-k-1)}^- w \rangle z^{-k-1} = \langle v, \psi_a^-(z^{-1})w \rangle z^{-1}.$$

4. FERMIONIC TRANSLATION OPERATORS AND TRANSLATION GROUP

Besides the action of fermion operators, ${}_a\psi_{(k)}^\pm$, we also have on $F^{(n)}$ the action of fermionic translation operators, $Q_a: F^{(n)} \rightarrow F^{(n)}$, $0 \leq a \leq n-1$, given by

$$Q_a v_0 = \psi_a^+(z) v_0 \mid_{z=0},$$

and

$$(3) \quad \psi_a^\pm(z) Q_a = z^{\pm 1} Q_a \psi_a^{\pm 1}(z),$$

$$(4) \quad \psi_a^\pm(z) Q_b = -Q_b \psi_a^{\pm 1}(z), \quad a \neq b,$$

$$(5) \quad Q_a Q_b = -Q_b Q_a, \quad a \neq b.$$

The Q_a are invertible. The $Q_a^{\pm 1}$ have degree $\pm\delta_a$.

The Q_a are unitary for the standard bilinear form of $F^{(n)}$:

$$(6) \quad \langle Q_a v, w \rangle = \langle v, Q_a^{-1} w \rangle, \quad 0 \leq a \leq n-1.$$

The group generated by Q_a , $0 \leq a \leq n-1$, contains a subgroup of elements of total degree zero, generated by the translation operators $T_i = Q_i Q_{i-1}^{-1}$ of degree $\delta_{i-1} - \delta_i$.

Lemma 4.1. (1) $T_a Q_b = Q_b T_a$ if $a \neq b, b+1$ and
 (2) $T_a Q_b = -Q_b T_a$ if $a = b, b+1$.

(3) For $m \in \mathbb{Z}$

$$T_i^m = (Q_i Q_{i-1}^{-1})^m = (-1)^{\frac{m(m-1)}{2}} Q_i^m Q_{i-1}^{-m}.$$

Proof. Parts (1), (2) are clear. Part (3) is a simple induction. \square

Define the ordered product of $k > 0$ fermions by

$$(7) \quad \prod_{\ell=1}^{\vec{k}} a \psi_{(-\ell)}^{\pm} = a \psi_{(-k)}^{\pm} \cdots a \psi_{(-1)}^{\pm}.$$

The empty product is as usual the identity.

Lemma 4.2. (1) For all $k \in \mathbb{Z}$, $k \neq 0$, $0 \leq a \leq n-1$,

$$Q_a^k v_0 = \begin{cases} v_0 & k = 0 \\ \prod_{\ell=1}^{\vec{k}} a \psi_{(-\ell)}^+ v_0 & k > 0, \\ \prod_{\ell=1}^{\vec{-k}} a \psi_{(-\ell)}^- v_0 & k < 0. \end{cases}$$

(2)

$$\begin{aligned} Q_1^\beta Q_0^\gamma v_0 &= (-1)^{\beta\gamma} \prod_{\ell=1}^{\vec{\gamma}} 0 \psi_{(-\ell)}^{\pm} \prod_{m=1}^{\vec{\beta}} 1 \psi_{(-m)}^{\pm} v_0 = \\ &= \prod_{m=1}^{\vec{\beta}} 1 \psi_{(-m)}^{\pm} \prod_{\ell=1}^{\vec{\gamma}} 0 \psi_{(-\ell)}^{\pm} v_0. \end{aligned}$$

\square

The translation operator T_i is also unitary, just as the fermionic translation operators: from (6) it follows that

$$(8) \quad \langle T_i v, w \rangle = \langle v, T_i^{-1} w \rangle.$$

5. THE LIE ALGEBRA $\mathfrak{gl}_\infty^{(n)}$

Define the Lie algebra $\mathfrak{gl}_\infty^{(n)}$ as the Lie subalgebra of $gl(H^{(n)})$ generated by $E_{a,b}^{k,l}$, $0 \leq a, b \leq n-1$, $k, l \in \mathbb{Z}$, where

$$E_{a,b}^{k,l} e_c z^m = \delta_{bc} \delta_{lm} e_a z^k.$$

Then $\mathfrak{gl}_\infty^{(n)}$ also acts on $F^{(n)}$, by

$$(9) \quad E_{a,b}^{k,l} \mapsto (e_a z^k \wedge)(i(e_b z^l)) = a \psi_{(k)b}^+ \psi_{(-l-1)}^-.$$

Introduce generating series for Lie algebra elements acting on $F^{(n)}$ by

$$(10) \quad E_{a,b}(z, w) = \sum_{k,l \in \mathbb{Z}} E_{a,b}^{k,l} z^{-k-1} w^l$$

We have an expression in terms of fermion fields for the generating series:

$$(11) \quad E_{a,b}(z, w) = \sum_{k,l \in \mathbb{Z}} a \psi_{(k)b}^+ \psi_{(-l-1)}^- z^{-k-1} w^l = \psi_a^+(z) \psi_b^-(w).$$

When $a = b$, we need the normal ordering. For the sake of completeness, this will be discussed below, but will not be required for any of our calculations.

We will need the commutator of the generating series of Lie algebra elements with fermionic translation operators, in a special case, see Lemma 5.1.

Lemma 5.1. *For all $\alpha, \beta \in \mathbb{Z}$ we have*

$$Q_0^\beta Q_1^{-\alpha} E_{1,0}(z, w) Q_1^\alpha Q_0^{-\beta} = (-1)^{\alpha+\beta} z^\alpha w^\beta E_{1,0}(z, w).$$

6. GAUSS FACTORIZATION AND FERMION MATRIX ELEMENTS

Let $GL_\infty^{(n)}$ be the group of invertible infinite matrices that differ from the identity by an element of $gl_\infty^{(n)}$. $GL_\infty^{(n)}$ acts on $H^{(n)} = \mathbb{C}^n \otimes \mathbb{C}[z, z^{-1}]$ and its central extension, $\widehat{GL}_\infty^{(n)}$ on fermionic Fock space $F^{(n)} = \Lambda^{\frac{\infty}{2}} H^{(n)}$ in the obvious way that is compatible with the $gl_\infty^{(n)}$ (resp. $\widehat{gl}_\infty^{(n)}$) action. The decomposition $H^{(n)} = H_+^{(n)} \oplus H_-^{(n)}$, where $H_+^{(n)} = \mathbb{C}^n \otimes \mathbb{C}[z]$ and $H_-^{(n)} = \mathbb{C}^n \otimes \mathbb{C}[z^{-1}]z^{-1}$, induces a block decomposition on elements $g \in GL_\infty^{(n)}$: every such g can be written as

$$g = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix},$$

where $A_{ij}: H_j^{(n)} \rightarrow H_i^{(n)}$.

We say that $g \in GL_\infty^{(n)}$ has a *Gauss Decomposition* if we can factor g as follows:

$$(12) \quad g = g_- g_{0+}, \quad g_- = \begin{pmatrix} 1 & 0 \\ X_{-+} & 1 \end{pmatrix}, \quad g_{0+} = \begin{pmatrix} A_{++} & A_{+-} \\ 0 & E_{--} \end{pmatrix}.$$

Let $\tau = \langle v_0, g v_0 \rangle$. g has a Gauss factorization if and only if $\tau \neq 0$. In this case ($\tau \neq 0$) we can give an explicit formula for the factor g_- of the Gauss factorization of g , in terms matrix elements of fermion insertions.

First we define the normal ordered product of fields. To this end decompose fermions in creation and annihilation fields²:

$$\psi^\pm(z) = \psi_{\text{cr}}^\pm(z) + \psi_{\text{ann}}^\pm(z),$$

where

$$\psi_{\text{cr}}^\pm(z) = \sum_{k \geq 0} \psi_{(-k-1)}^\pm z^k, \quad \psi_{\text{ann}}^\pm(z) = \sum_{k \geq 0} \psi_{(k)}^\pm z^{-k-1}.$$

Then the *Normal Ordered Product* is defined by

$$: \psi^*(z) \psi^\circ(w) : = \psi_{\text{cr}}^*(z) \psi^\circ(w) - \psi^\circ(w) \psi_{\text{ann}}^*(z),$$

where $*, \circ \in \{0, 1, \dots, n-1\}$ (independently) particular we have

$$(13) \quad : \psi^+(z) \psi^-(w) : v_0 = \psi_{\text{cr}}^+(z) \psi_{\text{cr}}^-(w) v_0.$$

Lemma 6.1. *Let $g \in GL_\infty^{(n)}$ have a Gauss factorization (so that $\tau \neq 0$) with negative component $g_- = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}$. Define*

$$g_{ab}(z, w) = \langle v_0, : \psi_b^+(w) \psi_a^-(z) : g v_0 \rangle / \tau.$$

Then

$$X = \sum_{a,b=0,1} \text{Res}_{z,w} (g_{ab}(z, w) E_{ab}(z, w)),$$

where $E_{ab}(z, w)$ is the generating series (10) of Lie algebra elements.

²We omit subscripts a on the fermions in this discussion of normal order.

Proof. Write $X = \sum_{a,b=0,1} X_{ab}$, where

$$X_{ab} = \sum_{r,s \geq 0} x_{ab}^{r,s} E_{ab}^{-r-1,s}.$$

Introduce generating series

$$x_{ab}(z, w) = \sum_{r,s} x_{ab}^{r,s} z^{-r-1} w^{-s-1}.$$

so that

$$X_{ab} = \text{Res}_{z,w}(x_{ab}(z, w) E_{ab}(z, w)).$$

Here

$$\text{Res}_{z,w} = \text{Res}_z \text{Res}_w,$$

and Res_z is the coefficient of z^{-1} in a series in z, z^{-1} .

Now we can calculate the coefficients $x_{ab}^{r,s}$ using the semi-infinite wedge space $F^{(n)}$. Consider $g_- v_0$. First note that

$$g_- v_0 = v_0 + \sum_{a,b} \sum_{r,s \geq 0} x_{ab}^{r,s} E_{ab}^{-r-1,s} v_0 + \dots,$$

where the omitted terms are quadratic and higher in E . Next note that if g has Gauss factorization $g_- v_0 = g v_0 / \tau$. Hence

$$x_{ab}^{r,s} = \langle E_{ab}^{-r-1,s} v_0, g_- v_0 \rangle = \langle E_{ab}^{-r-1,s} v_0, g v_0 \rangle / \tau.$$

We then get the generating series by (9) and (11)

$$\begin{aligned} x_{ab}(z, w) &= \sum_{r,s \geq 0} z^{-r-1} w^{-s-1} \langle a \psi_{(-r-1)}^+ b \psi_{(-s-1)}^- v_0, g v_0 \rangle / \tau = \\ &= z^{-1} w^{-1} \langle : \psi_a^+(z^{-1}) \psi_b^-(w^{-1}) : v_0, g v_0 \rangle / \tau. \end{aligned}$$

Then using the adjointness property (2) we move the normal ordered product to the other side to find

$$x_{ab}(z, w) = \langle v_0, : \psi_b^+(w) \psi_a^-(z) : g v_0 \rangle / \tau,$$

i.e., $x_{ab}(z, w) = g_{ab}(z, w)$ as we wanted to show. \square

We note that all of our calculations involve computing only $g_{ab}(z, w)$ for $a \neq b$, so in what follows, the normal ordering can be effectively ignored.

7. ROOT LATTICE

Recall the group \mathbb{Z}^n that gives a grading for fermionic Fock space $F^{(n)}$. It contains as a subgroup the root lattice A_{n-1} , generated by

$$\alpha_i = \delta_{i-1} - \delta_i, \quad 1 \leq i \leq n-1$$

So

$$A_{n-1} = \bigoplus_{i=1}^{n-1} \mathbb{Z} \alpha_i \subset \mathbb{Z}^n.$$

We will call elements in A_{n-1} of the form $\alpha = \sum n_i \alpha_i$ positive roots if all $n_i \geq 0$.

The translation group is also graded by A_{n-1} : as was mentioned earlier, the generator $T_i = Q_i Q_{i-1}^{-1}$ has degree α_i . Similarly the Lie algebra generating field $E_{ii-1}(z, w)$ has $\deg \alpha_i$.

8. A LOWER TRIANGULAR SUBGROUP

Consider in $\text{End}(H^{(n)})$ the element

$$g = 1_{H^{(n)}} + \sum_{a>b} \sum_{k,\ell \in \mathbb{Z}} c_{k,\ell}^{a,b} E_{a,b}^{-k-1,\ell}.$$

Here the $c_{k,\ell}^{a,b}$ are formal variables. Since $E_{a,b}^{k,\ell} E_{a,b}^{m,n} = 0$ (for all $k, \ell, m, n \in \mathbb{Z}$), the matrix g is invertible. We have the following lemma which will be useful later:

Lemma 8.1. *Let $g_-^{[k_1, \dots, k_{n-1}](\beta_0, \dots, \beta_{n-1})}$ be the negative part of the Gauss factorization of*

$$T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1} g^{(\beta_0, \dots, \beta_{n-1})},$$

where $T_i = Q_i Q_{i-1}^{-1}$, $g^{(\beta_0, \beta_1, \dots, \beta_{n-1})}$ are the elements in $GL_\infty^{(n)}$, the non-centrally extended infinite matrix group. Here, Q_i denotes the projection of the corresponding fermionic translation operator onto $GL_\infty^{(n)}$ and $g^{(\beta_0, \beta_1, \dots, \beta_{n-1})}$ is the group element obtained from (the projection of) g by conjugating by $Q_0^{\beta_0} \dots Q_{n-1}^{\beta_{n-1}}$, i.e.,

$$g^{(\beta_0, \dots, \beta_{n-1})} = Q_0^{\beta_0} \dots Q_{n-1}^{\beta_{n-1}} g Q_{n-1}^{-\beta_{n-1}} \dots Q_0^{-\beta_0} \in GL_\infty^{(n)}.$$

Then the following are nonnegative, i.e., they contain no terms of the form $E_{ab}^{-i-1,j}$ where $i, j \geq 0$.

(1)

$$(g_-^{[k_1, \dots, k_{n-1}](\beta_0, \beta_1, \dots, \beta_{n-1})})^{-1} Q_i^{-1} g_-^{[k_1, \dots, k_{n-1}](\beta_0, \dots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \dots, \beta_{n-1})},$$

where $0 \leq i \leq n-1$.

(2)

$$(g_-^{[k_1, \dots, k_{n-1}](\beta_0, \beta_1, \dots, \beta_{n-1})})^{-1} Q_{i-1}^{-1} g_-^{[k_1, \dots, k_i, k_i+1, \dots, k_{n-1}](\beta_0, \dots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \dots, \beta_{n-1})},$$

where $1 \leq i \leq n-1$.

(3)

$$(g_-^{[k_1, \dots, k_{n-1}](\beta_0, \beta_1, \dots, \beta_{n-1})})^{-1} Q_i^{-1} g_-^{[k_1, \dots, k_{i-1}, k_i-1, k_i, \dots, k_{n-1}](\beta_0, \dots, \beta_{i-2}, \beta_{i-1}+1, \beta_i, \dots, \beta_{n-1})},$$

where $1 \leq i \leq n-1$.

Proof. We prove only (1) here since the proofs of (2) and (3) are similar. We remind the reader that all calculations here are occurring in the non-centrally extended group, $GL_\infty^{(n)}$. For (1), we have

$$Q_i g^{(\beta_0, \dots, \beta_{n-1})} Q_i^{-1} = g^{(\beta_0, \dots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \dots, \beta_{n-1})},$$

so

$$\begin{aligned} Q_i T_{n-1}^{-k_{n-1}} T_{n-2}^{-k_{n-2}} \dots T_1^{-k_1} g^{(\beta_0, \dots, \beta_{n-1})} &= Q_i g_-^{[k_1, \dots, k_{n-1}](\beta_0, \dots, \beta_{n-1})} g_{0+}^{[k_1, \dots, k_{n-1}](\beta_0, \dots, \beta_{n-1})} = \\ &= g_-^{[k_1, \dots, k_{n-1}](\beta_0, \dots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \dots, \beta_{n-1})} g_{0+}^{[k_1, \dots, k_{n-1}](\beta_0, \dots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \dots, \beta_{n-1})} Q_i \end{aligned}$$

So

$$\begin{aligned} (g_-^{[k_1, \dots, k_{n-1}](\beta_0, \beta_1, \dots, \beta_{n-1})})^{-1} Q_i^{-1} g_-^{[k_1, \dots, k_{n-1}](\beta_0, \dots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \dots, \beta_{n-1})} &= \\ &= g_{0+}^{[k_1, \dots, k_{n-1}](\beta_0, \beta_1, \dots, \beta_{n-1})} Q_i^{-1} (g_{0+}^{[k_1, \dots, k_{n-1}](\beta_0, \dots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \dots, \beta_{n-1})})^{-1}. \end{aligned}$$

□

Remark 8.2. A corollary of Lemma 8.1 is that the $\widehat{GL}_\infty^{(n)}$ tau-functions, for $n \geq 3$, satisfy bilinear relations with only three terms. See Section 11.1 for the derivation of three term relations for $n = 3$. The same process can be followed for any $n \geq 3$. An analogous lemma for the \widehat{GL}_n case shows that these tau-functions also satisfy bilinear relations with three terms.

9. $\widehat{GL}_\infty^{(2)}$ CASE

We now specialize to the case $n = 2$. We can write

$$(14) \quad g = \exp(\Gamma), \quad \Gamma = \sum_{k,l \geq 0} c_{k,l} E_{1,0}^{-k-1,l}.$$

In this form g acts also on fermionic Fock space $F^{(2)}$. If we write g^c to make the dependence on the $c_{k,l}$ more explicit we have $g^c g^d = g^{c+d}$ and these elements form a subgroup N of $GL(H^{(n)})$ and $GL(F^{(n)})$. We can also think of the $c_{k,l}$ as coordinates on this subgroup N .

It will be useful to collect the $c_{k,l}$ in a generating series:

$$(15) \quad C(z, w) = \sum_{k,l \geq 0} c_{k,l} z^{-k-1} w^{-l-1}$$

so that

$$(16) \quad \Gamma = \text{Res}_{z,w} C(z, w) E_{1,0}(z, w).$$

Since $\deg(\Gamma)$ has degree α , i.e., is positive, we see that gv_0 is a sum of elements with degree a positive root in A_1 .

Let $B = \mathbb{C}[c_{k,l}]_{k,l \geq 0}$ be the coordinate ring of the group N . We describe some structures related to B .

First define shifts acting on B : these are multiplicative maps (on generators)

$$(17) \quad S^{\alpha,\beta}: B \rightarrow B, \quad c_{k,l} \mapsto c_{k+\alpha, l+\beta}, \quad \alpha, \beta \in \mathbb{Z}, \quad \alpha, \beta \geq 0, \quad S^{\alpha,\beta}(1) = 0.$$

Define also *shift fields*. These are multiplicative maps

$$(18) \quad S^{\pm 1,0}(y), S^{0,\pm 1}(y): B \rightarrow B[[y^{-1}]], \quad a = 0, 1$$

given by

$$(19) \quad S^{\pm 1,0}(y) = (1 - \frac{S^{1,0}}{y})^{\pm 1}, \quad S^{0,\pm 1}(y) = (1 - \frac{S^{0,1}}{y})^{\pm 1}.$$

Recall the Lie algebra element Γ from (16). The coefficients of Γ belong to B , so we can apply shifts to Γ coefficient wise. Define

$$\Gamma^{\alpha,\beta} = S^{\alpha,\beta} \Gamma = \sum_{k,l \geq 0} c_{k+\alpha, l+\beta} E_{10}^{-k-1,l}.$$

We can relate the shift in Γ acting on the vacuum $v_0 \in F^{(n)}$ in terms of conjugation by fermionic translation operators, using Lemma 5.1.

Lemma 9.1.

- (1) $E_{10}^{-k-1,l} v_0 = 0$, if $k < 0$, or $l < 0$ (as long as $l \neq -k - 1$).

(2)

$$(S^{\alpha,\beta}\Gamma)v_0 = (-1)^{\alpha+\beta}Q_0^\alpha Q_1^{-\beta}\Gamma Q_1^\beta Q_0^{-\alpha}v_0.$$

Proof. For Part (1) we have $E_{10}^{-k-1,l} = (e_1 z^{k+l} \wedge)(i(e_0 z^\ell))$. Now for $l < 0$ v_0 does not contain a wedge factor $e_0 z^\ell$ so that $i(e_0 z^\ell)v_0 = 0$.

For Part (2) write

$$(S^{\alpha,\beta}\Gamma)v_0 = \sum_{k,\ell \in \mathbb{Z}} c_{k+\alpha,\ell+\beta} E_{10}^{-k-1,l} v_0,$$

which can be written as

$$(S^{\alpha,\beta}\Gamma)v_0 = \text{Res}_{z_1, w_1} \left(C(z_1, w_1) z_1^\alpha w_1^\beta E_{10}(z_1, w_1) \right).$$

Then the lemma follows from Lemma 5.1. \square

9.2. τ -functions. We will define an infinite collection of elements of B by

$$(20) \quad \tau_k(g) = \langle T^k v_0, g v_0 \rangle.$$

Lemma 9.2. $\tau_k(g)$ is zero if $k < 0$.

Proof. We saw in Section 7 that $g v_0$ is a sum of terms of degree a positive root in A_1 . Now $T^k v_0$ has degree $k\alpha$, which is positive if and only if $k \geq 0$. The Lemma follows then from orthogonality of terms of different degree in $F^{(n)}$. \square

Our aim is to give formulas for $\tau_k(g)$. First note that we can expand the exponential in $g = \exp(\Gamma)$, see (61). We get

$$\tau_k(g) = \sum_{l \geq 0} \langle T^k v_0, \frac{\Gamma^l}{l!} v_0 \rangle.$$

Now T^k has degree $k\alpha$ and similarly Γ^l has degree $l\alpha$. Now the homogeneous subspaces of $F^{(n)}$ of different bidegree are orthogonal for the bilinear form. This means that the only non zero contributions to the sum are those where $k = l$. In other words

$$(21) \quad \tau_k(g) = \langle T^k v_0, \frac{\Gamma^k}{k!} v_0 \rangle.$$

9.3. Conditions. Recall the series $C(z, w)$ appearing in the group element $g = \exp(\Gamma)$, see (15). We can use these series to define a linear map (“condition”) on polynomials:

$$(22) \quad c: \mathbb{C}[z, w] \rightarrow \mathbb{C}, \quad f(z, w) \mapsto \text{Res}_{z, w}(C(z, w)f(z, w)).$$

In particular the coefficients of $z^{-k-1}w^{-l-1}$ in $C(z, w)$ is a moment of the condition c : we have $c_{k,l} = c(z^k w^l)$, so that

$$(23) \quad C(z, w) = \sum_{k,l \geq 0} c_1(z_1^k w_1^l) z^{-k-1} w^{-l-1} = c_1 \left(\frac{1}{z - \underline{z_1}} \frac{1}{w - \underline{w_1}} \right).$$

In terms of the condition c the shift fields $S^{\pm 1,0}(y)$, $S^{0,\pm 1}(y)$ correspond to insertion of a factor $(1 - \frac{z}{y})^{\pm 1}$ or $(1 - \frac{w}{y})^{\pm 1}$ in the generators of B :

$$(24) \quad \begin{aligned} S^{\pm 1,0}(y)c_{k,l} &= c(z^k w^l (1 - \frac{z}{y})^{\pm 1}) = y^{\mp 1} c(z^k w^l (y - \underline{z})^{\pm 1}), \\ S^{0,\pm 1}(y)c_{k,l} &= c(z^k w^l (1 - \frac{w}{y})^{\pm 1}) = y^{\mp 1} c(z^k w^l (y - \underline{w})^{\pm 1}). \end{aligned}$$

We will need not just conditions on polynomials in a single set of variables z, w , but also conditions on polynomials in multiple sets of variables z_i, w_i . First define multiple copies of the single condition:

$$c_i : \mathbb{C}[z_i, w_i] \rightarrow \mathbb{C}, \quad z_i^k w_i^l \mapsto c(z^k w^l) = c_{k,l}.$$

Then define product conditions (with ϵ a positive integer)

$$\prod_{i=1}^{\epsilon} c_i : \mathbb{C}[z_1, w_1, z_2, w_2, \dots, z_{\epsilon}, w_{\epsilon}] \rightarrow \mathbb{C}$$

on monomials by

$$\prod_{i=1}^{\epsilon} c_i \left(\prod_{j=1}^{\epsilon} z_j^{d_j} w_j^{e_j} \right) = \prod_{i=1}^{\epsilon} c_{d_i, e_i},$$

and extend by linearity.

We will extend our conditions (or product of conditions) from polynomials to series. So we can write

$$\Gamma = \text{Res}_{z,w}(C(z, w)E_{ab}(z, w)) = c(E_{ab}(z, w)),$$

where we use that $c(z^k w^l) = 0$ if $k < 0$ or $l < 0$. More generally

$$\Gamma^{\epsilon} = \prod_{i=1}^{\epsilon} c_i \left(\prod_{j=1}^{\epsilon} E_{10}(z_j, w_j) \right).$$

9.4. Fermion Correlation Functions. Introduce³

$$(25) \quad \Gamma_k^k(z_1, w_1, z_2, w_2, \dots, z_k, w_k) = \langle T^k v_0, \prod_{j=1}^k \frac{E_{10}(z_j, w_j)}{k!} v_0 \rangle,$$

so that

$$\tau_k(g) = \prod_{i=1}^k c_i(\Gamma_k^k(z_1, w_1, z_2, w_2, \dots, z_k, w_k)).$$

It will turn out that the Γ_k^k are products of Vandermonde determinants in the variables z_i, w_i .

Note that

$$\Gamma_k^k(z_1, w_1, z_2, w_2, \dots, z_k, w_k) = \langle T^k v_0, \prod_{j=1}^k \psi_1^+(z_j) \psi_0^-(w_j) v_0 \rangle,$$

so that to calculate $\tau_k(g)$ we need to calculate the fermion correlation functions Γ_k^k .

³The product is well defined because the $E_{1,0}(z_i, w_i)$ commute.

9.5. Factorization and Reduction to One Component Fermions. Suppose we want to calculate a matrix element of fermion fields of the form

$$\langle Q_1^\alpha Q_0^\beta v_0, F(\psi_a^\pm(z)) v_0 \rangle,$$

where F is some polynomial in the fermion fields. By linearity we can reduce to the case where $F = M$ is a monomial, and then we can in the monomial rearrange the factors as in Definition 2.1: $M = M_1 M_0$, $M_a = M_a^+ M_a^-$, where M_a^\pm is a monomial in a single type of fermions, ordered according to the subscript of the arguments of the fields:

$$(26) \quad M_a^\pm = \psi_a^\pm(z_\epsilon) \psi_a^\pm(z_{\epsilon-1}) \dots \psi_a^\pm(z_1) = \prod_{i=1}^{\vec{\epsilon}} \psi_a^\pm(z_i).$$

This defines the ordered product of fermion fields. (This is different from the ordering of the components given by (7).)

We calculate such matrix elements using the following factorization lemma.

Lemma 9.3. *Let $M_a = M_a(\psi_a^\pm(z))$, $a = 0, 1$, be monomials in fermion fields of type a . Then*

$$\langle Q_1^\alpha Q_0^\beta v_0, M_1 M_0 v_0 \rangle = \langle Q_1^\alpha, M_1 v_0 \rangle \langle Q_0^\beta, M_0 v_0 \rangle.$$

So the correlation functions we want to calculate reduce to products of correlation functions containing only one type of fermionic translation operator Q_a and one type of fermion fields $\psi_a^\pm(z)$. Such correlation functions (on two component semi-infinite wedge space) are the same as the corresponding correlation function calculated on F , one component semi-infinite wedge space, according to the following Lemma.

Lemma 9.4.

$$\langle Q_a^\alpha v_0, F(\psi_a^\pm) v_0 \rangle_{F^{(n)}} = \langle Q^\alpha v_0, F(\psi^\pm) v_0 \rangle_F.$$

Lemma 9.5. $\langle Q^{\pm k} v_0, \prod_{i=1}^{\vec{k}} \psi^\pm(z_i) v_0 \rangle = \prod_{1 \leq i < j \leq k} (z_j - z_i).$

For a proof of this, see [AB].

10. HEINE FORMULAS

Lemma 10.1. *For $k \in \mathbb{C}, k > 0$ we have*

$$\Gamma_k^k = \langle T^k v_0, \prod_{i=1}^k E_{1,0}(z_i, w_i) v_0 \rangle = \det \left(V_{\{z_i\}}^{(k)} V_{\{w_i\}}^{(k)} \right),$$

where the Vandermonde matrix of size $k \times k$ is

$$(27) \quad V_{\{z_i\}}^{(k)} = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_k \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{k-1} & z_2^{k-1} & \dots & z_k^{k-1} \end{pmatrix}.$$

Proof. We have $T = Q_1 Q_0^{-1}$ and similarly $E_{1,0}(z, w)$ is a product of $\psi_1^+(z)$ and $\psi_0^-(w)$. By the factorization Lemma 9.3

$$\langle T^k v_0, \prod_{i=1}^k E_{1,0}(z_i) v_0 \rangle = \langle Q_1^k v_0, \prod_{i=1}^{\vec{k}} \psi_1^+(z_i) v_0 \rangle \langle Q_0^{-k} v_0, \prod_{i=1}^{\vec{k}} \psi_0^-(w_i) v_0 \rangle.$$

The lemma then follows from the 1-component fermion case, Lemma 9.5. \square

In particular for $k = 1$ we get $\Gamma_1^1 = \langle Tv_0, E_{1,0}(z, w)v_0 \rangle = 1$.

Corollary 10.2.

$$\tau_k(g) = \det \begin{pmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,k-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k-1,0} & c_{k-1,1} & \cdots & c_{k-1,k-1} \end{pmatrix}$$

Proof. We have

$$\tau_k(g) = \langle T^k v_0, \frac{\Gamma^k}{k!} v_0 \rangle,$$

where $\Gamma = \text{Res}_{z,w}(C(z, w)E_{1,0}(z, w))$. Then, using the notation of Section 9.3,

$$\langle T^k v_0, \Gamma^k v_0 \rangle = \prod_{i=1}^k c_i(\langle T^k v_0, \prod_{j=1}^k E_{1,0}(z_j, w_j)v_0 \rangle) = \prod_{i=1}^k c_i(\det(V_{\{z_i\}}^{(k)} \det(V_{\{w_i\}}^{(k)})),$$

by Lemma 10.1. Expanding one of $\det V_{\{z_i\}}^{(k)}$ using the Leibniz form of the determinant and writing the other as in (27), we see that

$$\prod_{i=1}^k c_i(\det(V_{\{z_i\}}^{(k)} \det(V_{\{w_i\}}^{(k)}))) = k! \det \begin{pmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,k-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k-1,0} & c_{k-1,1} & \cdots & c_{k-1,k-1} \end{pmatrix}$$

and the result follows. \square

10.1. $\widehat{GL}_\infty^{(2)}$ **Tau-Functions Satisfy T -system Relations.** For clarity, we denote elements of the non-centrally extended group with a bar. We will later on drop the bar, expecting the reader to decide for herself when we are in the central extension or not.

Definition 10.3. Let $\overline{g_C} \in GL_\infty^{(2)}$ be given by $\overline{g} = I + \sum_{i,j \in \mathbb{Z}} c_{ij} E_{10}^{-i-1,j}$ where $c_{i,j}$ are complex numbers or formal variables and I denotes the identity.

We also need to define shifted group elements, $\overline{g_C^{(\alpha,\beta)}}$:

Definition 10.4. $\overline{g_C^{(\alpha,\beta)}} = \overline{Q_0}^{-\alpha} \overline{Q_1}^{-\beta} \overline{g_C} \overline{Q_1}^{-\beta} \overline{Q_0}^{-\alpha}$ where $\overline{Q_0} = \sum_{i \in \mathbb{Z}} E_{00}^{i-1,i} + \sum_{i \in \mathbb{Z}} E_{11}^{i,i}$ and $\overline{Q_1} = \sum_{i \in \mathbb{Z}} E_{00}^{i,i} + \sum_{i \in \mathbb{Z}} E_{11}^{i-1,i}$:

$$\begin{aligned} \overline{Q_0} \overline{g_C} \overline{Q_0}^{-1} &= \left(\sum_{i \in \mathbb{Z}} E_{00}^{i,i+1} + \sum_{i \in \mathbb{Z}} E_{11}^{i,i} \right) \times \\ &\quad \left(I + \sum_{i,j \in \mathbb{Z}} c_{ij} E_{10}^{-i-1,j} \right) \left(\sum_{i \in \mathbb{Z}} E_{00}^{i,i+1} + \sum_{i \in \mathbb{Z}} E_{11}^{i,i} \right) = \\ &= I + \sum_{i,j \in \mathbb{Z}} c_{i,j} E_{10}^{-i-1,j-1} = I + \sum_{i,j \in \mathbb{Z}} c_{i,j+1} E_{10}^{-i-1,j}. \end{aligned}$$

So conjugating by $\overline{Q_0}$ shifts the j index up by one. Similarly, conjugating $\overline{g_C}$ by $\overline{Q_1}$ shifts the i index down by 1.

Definition 10.5. Let $T = Q_1 Q_0^{-1}$.

The $\widehat{GL}_\infty^{(2)}$ tau-functions are the following matrix elements:

Definition 10.6. $\tau_k^{(\alpha, \beta)} = \langle T^k v_0, g^{(\alpha, \beta)} v_0 \rangle$

Here, $g^{(\alpha, \beta)}$ denotes a lift to $\widehat{GL}_\infty^{(2)}$ of the group element given in Definition 10.4, and tau-functions are determined up to a constant.

Theorem 10.7. $\tau_k^{(\alpha, \beta)} =$

$$\det \begin{pmatrix} c_{-\beta, \alpha} & c_{-\beta+1, \alpha} & \cdots & c_{-\beta+k-1, \alpha} \\ c_{-\beta, \alpha+1} & c_{-\beta+1, \alpha+1} & \cdots & c_{-\beta+k-1, \alpha+1} \\ \vdots & \vdots & \cdots & \vdots \\ c_{-\beta, \alpha+k-1} & c_{-\beta+1, \alpha+k-1} & \cdots & c_{-\beta+k-1, \alpha+k-1} \end{pmatrix}$$

Proof. The proof of this follows directly from the definition of $g_C^{(\alpha, \beta)}$ and Corollary 10.2. \square

We will compute $g_{ab}(z, w)$ for $a, b = 0, 1$ and $g = T^{-k} g_C$. For now, we will ignore the shifts in α and β , since here they only serve to complicate our notation.

Lemma 10.8.

$$(28) \quad g_{01}(z, w) = \frac{S^{+1,0}(w) S^{0,+1}(z) \tau_{k-1}}{w z \tau_k}$$

$$(29) \quad g_{10}(z, w) = \frac{S^{-1,0}(z) S^{0,-1}(w) \tau_{k+1}}{w z \tau_k}$$

where $S^{\pm,0}(z)$ are the shift fields defined by (19).

Theorem 10.9. The $\tau_k^{(\alpha, \beta)}$ satisfy the following difference equations, for all $k \geq 0$ and $\alpha, \beta \in \mathbb{Z}$:

$$(2T) \quad \tau_{k+1}^{(\alpha, \beta+1)} \tau_{k-1}^{(\alpha+1, \beta)} = \tau_k^{(\alpha+1, \beta)} \tau_k^{(\alpha, \beta+1)} - \tau_k^{(\alpha, \beta)} \tau_k^{(\alpha+1, \beta+1)}$$

Proof. We can obtain this by applying the Desnanot-Jacobi identity, or as follows:

By Lemma 8.1,

$$(g_-^{[k](\alpha, \beta)})^{-1} Q_0^{-1} g_-^{[k](\alpha+1, \beta)} \text{ and } (g_-^{[k](\alpha, \beta)})^{-1} Q_0^{-1} g_-^{[k+1](\alpha, \beta+1)}$$

are nonnegative. (Recall that nonnegative terms are linear combinations of $E_{ab}^{-i-1, j}$ where one or both of the i, j are negative. Note that this is the analogue of “nonnegative in z ” that we saw in the \widehat{GL}_2 and \widehat{GL}_3 cases in [AB].)

Denote $h_{ab}^{i, j, [k](\alpha, \beta)}$ to be the coefficient of $E_{ab}^{-i-1, j}$ in $g_-^{[k](\alpha, \beta)}$ for $(i, j) \neq (0, 0)$ and denote $h_{ab}^{[k](\alpha, \beta)} = h_{ab}^{0, 0, [k](\alpha, \beta)}$, the coefficient $E_{ab}^{-1, 0}$ in $g_-^{[k](\alpha, \beta)}$. Then

$$\begin{aligned} & (g_-^{[k](\alpha, \beta)})^{-1} Q_0^{-1} g_-^{[k](\alpha+1, \beta)} = \\ & (I - \sum_{i, j \geq 0} h_{ab}^{i, j, [k](\alpha, \beta)}) (\sum_{i \in \mathbb{Z}} E_{00}^{i+1, i} + \sum E_{11}^{ii}) (I + \sum_{i, j \geq 0} h_{ab}^{i, j, [k](\alpha+1, \beta)}) \end{aligned}$$

Since $(g_-^{[k](\alpha,\beta)})^{-1}Q_0^{-1}g_-^{[k+1](\alpha,\beta+1)}$ is nonnegative, the coefficient of $E_{11}^{-1,0}$ must be zero and therefore,

$$(30) \quad -h_{10}^{[k](\alpha,\beta)}h_{01}^{[k](\alpha+1,\beta)} - h_{11}^{[k](\alpha,\beta)} + h_{11}^{[k](\alpha+1,\beta)} = 0.$$

We can read off $h_{10}^{[k](\alpha,\beta)} = \frac{\tau_{k+1}^{(\alpha,\beta)}}{\tau_k^{(\alpha,\beta)}}$ and $h_{01}^{[k](\alpha+1,\beta)} = \frac{\tau_{k-1}^{(\alpha+1,\beta)}}{\tau_k^{(\alpha+1,\beta)}}$ from Lemma 10.8. So

$$(31) \quad h_{11}^{[k](\alpha+1,\beta)} - h_{11}^{[k](\alpha,\beta)} = \frac{\tau_{k+1}^{(\alpha,\beta)}\tau_{k-1}^{(\alpha+1,\beta)}}{\tau_k^{(\alpha,\beta)}\tau_k^{(\alpha+1,\beta)}}.$$

Similarly, the nonnegativity of $(g_-^{[k](\alpha,\beta)})^{-1}Q_0^{-1}g_-^{[k+1](\alpha,\beta+1)}$ gives

$$(32) \quad h_{11}^{[k+1](\alpha,\beta+1)} - h_{11}^{[k](\alpha,\beta)} = \frac{\tau_{k+1}^{(\alpha,\beta)}\tau_k^{(\alpha,\beta+1)}}{\tau_k^{(\alpha,\beta)}\tau_{k+1}^{(\alpha,\beta+1)}}.$$

The trivially satisfied relation

$$\begin{aligned} & (h_{11}^{[k](\alpha+1,\beta)} - h_{11}^{[k](\alpha,\beta)}) + (h_{11}^{[k+1](\alpha+1,\beta+1)} - h_{11}^{[k](\alpha+1,\beta)}) = \\ & = (h_{11}^{[k+1](\alpha,\beta+1)} - h_{11}^{[k](\alpha,\beta)}) + (h_{11}^{[k+1](\alpha+1,\beta+1)} - h_{11}^{[k+1](\alpha,\beta+1)}), \end{aligned}$$

by the above identities, is equivalent to

$$(33) \quad \frac{\tau_{k+1}^{(\alpha,\beta)}\tau_{k-1}^{(\alpha+1,\beta)}}{\tau_k^{(\alpha,\beta)}\tau_k^{(\alpha+1,\beta)}} + \frac{\tau_{k+1}^{(\alpha+1,\beta)}\tau_k^{(\alpha+1,\beta+1)}}{\tau_k^{(\alpha+1,\beta)}\tau_{k+1}^{(\alpha+1,\beta+1)}} = \frac{\tau_{k+1}^{(\alpha,\beta)}\tau_k^{(\alpha,\beta+1)}}{\tau_k^{(\alpha,\beta)}\tau_{k+1}^{(\alpha,\beta+1)}} + \frac{\tau_{k+2}^{(\alpha,\beta+1)}\tau_k^{(\alpha+1,\beta+1)}}{\tau_{k+1}^{(\alpha,\beta+1)}\tau_{k+1}^{(\alpha+1,\beta+1)}}.$$

Writing all terms under a common denominator, we obtain

$$(34) \quad \tau_{k+1}^{(\alpha,\beta)}\tau_{k+1}^{(\alpha+1,\beta+1)}(\tau_{k+1}^{(\alpha,\beta+1)}\tau_{k-1}^{(\alpha+1,\beta)} - \tau_k^{(\alpha+1,\beta)}\tau_k^{(\alpha,\beta+1)}) = \\ = \tau_k^{(\alpha,\beta)}\tau_k^{(\alpha+1,\beta+1)}(\tau_{k+2}^{(\alpha,\beta+1)}\tau_k^{(\alpha+1,\beta)} - \tau_{k+1}^{(\alpha+1,\beta)}\tau_{k+1}^{(\alpha,\beta+1)}).$$

Now if Theorem 10.9 holds for some k , (34) implies that it holds for $k+1$. But $\tau_{-1}^{(\alpha,\beta)} = 0$ and $\tau_0^{(\alpha,\beta)} = 1$ for all α and β , so Theorem 10.9 holds trivially for $k=0$. By induction, we obtain our result. \square

10.2. Proof of \widehat{GL}_2 Difference Relations. We can rederive the fact that our \widehat{GL}_2 tau-functions satisfy Q -system relations in the same way we derived the T -system relations above. As is stated and proved in [AB],

Theorem 10.10. *The \widehat{GL}_2 tau-functions satisfy*

$$(2Q) \quad (\tau_k^{(\alpha)})^2 = \tau_k^{(\alpha-1)}\tau_k^{(\alpha+1)} - \tau_{k+1}^{(\alpha-1)}\tau_{k-1}^{(\alpha+1)}$$

where $k = 0, 1, 2, \dots$ and $\alpha \in \mathbb{Z}$.

Proof. We proved in our paper that the following elements are nonnegative in z , i.e., their expressions contain no negative powers of z :

$$\begin{aligned} V_k^{(\alpha)} &= (g_-^{[k](\alpha)})^{-1}Q_0^{-1}g_-^{[k](\alpha+1)} \\ W_k^{(\alpha)} &= (g_-^{[k](\alpha)})^{-1}Q_1^{-1}g_-^{[k-1](\alpha+1)}. \end{aligned}$$

We note that the $g_-^{[k](\alpha)}$ s have determinant 1. We use the following two basic facts: The inverse of a two by two matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant one is $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and given an invertible matrix, $I + \frac{A}{z} + \mathcal{O}(z)$, its inverse is $I - \frac{A}{z} + \mathcal{O}(z)$.

Let E_{ab} be the 2×2 matrix with 1 in the a, b entry and 0 elsewhere and let $h_{ab}^{i, [k](\alpha)}$ denote the coefficient of $E_{ab} z^{-i-1}$ in $g_-^{[k](\alpha)}$ and let $h_{ab}^{[k](\alpha)} = h_{ab}^{0, [k](\alpha)}$. So

$$\begin{aligned} V_k^{(\alpha)} &= (I + \sum_{\substack{i=0 \\ a,b=0,1}}^{\infty} h_{ab}^{i, [k](\alpha)} E_{ab} z^{-i-1})^{-1} (E_{00} z + E_{11}) (I + \sum_{\substack{i=0 \\ a,b=0,1}}^{\infty} h_{ab}^{i, [k](\alpha+1)} E_{ab} z^{-i-1}) = \\ &= (I + h_{00}^{i, [k](\alpha)} E_{11} z^{-i-1} + h_{11}^{i, [k](\alpha)} E_{00} z^{-i-1} - h_{10}^{i, [k](\alpha)} E_{10} z^{-i-1} - h_{01}^{i, [k](\alpha)} E_{01} z^{-i-1}) \times \\ &\quad (E_{00} z + E_{11}) (I + \sum_{\substack{i=0 \\ a,b=0,1}}^{\infty} h_{ab}^{i, [k](\alpha+1)} E_{ab} z^{-i-1}). \end{aligned}$$

Since $V_k^{(\alpha)}$ is nonnegative in z , the coefficient of $E_{11} z^{-1}$ in this expression is 0. Therefore, we have

$$(35) \quad h_{00}^{[k](\alpha)} + h_{11}^{[k](\alpha+1)} - h_{10}^{[k](\alpha)} h_{01}^{[k](\alpha+1)} = 0.$$

But $h_{10}^{[k](\alpha)}$ and $h_{01}^{[k](\alpha+1)}$ are easily read off from the formula for the $g_-^{[k](\alpha)}$ s (see [AB]):

$$h_{10}^{[k](\alpha)} = \frac{\tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha)}} \text{ and } h_{01}^{[k](\alpha+1)} = \frac{\tau_{k-1}^{(\alpha+1)}}{\tau_k^{(\alpha+1)}}.$$

So

$$h_{00}^{[k](\alpha)} + h_{11}^{[k](\alpha+1)} = \frac{\tau_{k+1}^{(\alpha)} \tau_{k-1}^{(\alpha+1)}}{\tau_k^{(\alpha)} \tau_k^{(\alpha+1)}}$$

Using that $(I + \frac{A}{z} + \mathcal{O}(z))^{-1} = I - \frac{A}{z} + \mathcal{O}(z)$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ when $ad - bc = 1$, we see that $h_{00}^{[k](\alpha)} = -h_{11}^{[k](\alpha)}$. So we obtain

$$(36) \quad -h_{11}^{[k](\alpha)} + h_{11}^{[k](\alpha+1)} = \frac{\tau_{k+1}^{(\alpha)} \tau_{k-1}^{(\alpha+1)}}{\tau_k^{(\alpha)} \tau_k^{(\alpha+1)}}.$$

Using the same argument as above, this time with $W_k^{(\alpha)}$, the coefficient of $E_{00} z^{-1}$ is 0 gives

$$h_{11}^{[k](\alpha)} + h_{00}^{[k-1](\alpha+1)} = h_{01}^{[k](\alpha)} h_{10}^{[k-1](\alpha+1)} = \frac{\tau_{k-1}^{(\alpha)} \tau_k^{(\alpha+1)}}{\tau_k^{(\alpha)} \tau_{k-1}^{(\alpha+1)}}.$$

So

$$(37) \quad h_{11}^{[k](\alpha)} - h_{11}^{[k-1](\alpha+1)} = \frac{\tau_{k-1}^{(\alpha)} \tau_k^{(\alpha+1)}}{\tau_k^{(\alpha)} \tau_{k-1}^{(\alpha+1)}}$$

$$\begin{aligned} (38) \quad h_{11}^{[k+1](\alpha)} - h_{11}^{[k](\alpha+1)} + h_{11}^{[k](\alpha+1)} - h_{11}^{[k](\alpha)} &= \\ &= h_{11}^{[k+1](\alpha)} - h_{11}^{[k+1](\alpha-1)} + h_{11}^{[k+1](\alpha-1)} - h_{11}^{[k](\alpha)}. \end{aligned}$$

Using (36) and (37), this gives

$$(39) \quad \frac{\tau_k^{(\alpha)} \tau_{k+1}^{(\alpha+1)}}{\tau_{k+1}^{(\alpha)} \tau_k^{(\alpha+1)}} + \frac{\tau_{k-1}^{(\alpha+1)} \tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha+1)} \tau_{k-1}^{(\alpha)}} = \frac{\tau_k^{(\alpha)} \tau_{k+2}^{(\alpha-1)}}{\tau_{k+1}^{(\alpha)} \tau_{k+1}^{(\alpha-1)}} + \frac{\tau_k^{(\alpha-1)} \tau_{k+1}^{(\alpha)}}{\tau_{k+1}^{(\alpha-1)} \tau_k^{(\alpha)}}.$$

Proceeding exactly as we did in the proof of Theorem 10.9, i.e. bringing all terms under the same denominator, using the fact that $\tau_{-1}^{(\alpha)} = 0$ and $\tau_0^{(\alpha)} = 1$ for all $\alpha \in \mathbb{Z}$, and induction, our result follows. \square

11. $\widehat{GL}_\infty^{(3)}$ CASE

11.1. $\widehat{GL}_\infty^{(3)}$ Tau-Functions and Relations That They Satisfy. We generalize the above $\widehat{GL}_\infty^{(2)}$ case to the $\widehat{GL}_\infty^{(3)}$, following much the same procedure as above.

Definition 11.1. Let $g_{C,D,E} \in GL_\infty^{(3)}$ be given by $g = I + \sum_{i,j \in \mathbb{Z}} c_{ij} E_{10}^{-i-1,j} + \sum_{i,j \in \mathbb{Z}} d_{ij} E_{20}^{-i-1,j} + \sum_{i,j \in \mathbb{Z}} e_{ij} E_{21}^{-i-1,j}$ where the c_{ij}, d_{ij}, e_{ij} are complex numbers or formal variables and I denotes the identity.

We also need to define shifted group elements, $g_{C,D,E}^{(\alpha,\beta,\gamma)}$:

Definition 11.2. $g_{C,D,E}^{(\alpha,\beta,\gamma)} = Q_0^\alpha Q_1^\beta Q_2^\gamma g_{C,D,E} Q_2^{-\gamma} Q_1^{-\beta} Q_0^{-\alpha}$ where $Q_0 = \sum_{i \in \mathbb{Z}} E_{00}^{i-1,i} + \sum_{i \in \mathbb{Z}} E_{11}^{i,i} + \sum_{i \in \mathbb{Z}} E_{22}^{i,i}$, $Q_1 = \sum_{i \in \mathbb{Z}} E_{00}^{i,i} + \sum_{i \in \mathbb{Z}} E_{11}^{i-1,i} + \sum_{i \in \mathbb{Z}} E_{22}^{i,i}$, and $Q_2 = \sum_{i \in \mathbb{Z}} E_{00}^{i,i} + \sum_{i \in \mathbb{Z}} E_{11}^{i,i} + \sum_{i \in \mathbb{Z}} E_{22}^{i-1,i}$.

Remark 11.3. As in the $\widehat{GL}_\infty^{(2)}$ case, conjugation here occurs in the non-centrally extended group, $GL_\infty^{(2)}$.

Definition 11.4. Let $T_1 = Q_1 Q_0^{-1}$ and $T_2 = Q_2 Q_1^{-1}$.

The $\widehat{GL}_\infty^{(3)}$ tau-functions are the following matrix elements:

Definition 11.5. $\tau_{k,\ell}^{(\alpha,\beta)} = \langle T_1^k T_2^\ell v_0, g_{C,D,E}^{(\alpha,\beta,\gamma)} v_0 \rangle$.

As in the $\widehat{GL}_\infty^{(2)}$ case, $g_{C,D,E}^{(\alpha,\beta,\gamma)}$ is really a lift to $\widehat{GL}_\infty^{(2)}$ of the group element given in Definition 11.2. The calculation of these tau-functions is nearly identical to the calculation given of the \widehat{GL}_3 tau-functions in [AB]. We have the following theorem:

Theorem 11.6. $\tau_{k,\ell}^{(\alpha,\beta,\gamma)} = \sum_{\substack{k=n_c+n_d \\ \ell=n_d+n_e}} c_{n_c,n_d,n_e}^{(\alpha,\beta,\gamma)}$ where

$$c_{n_c,n_d,n_e}^{(\alpha,\beta,\gamma)} = \frac{(-1)^{\frac{n_d(n_d+1)}{2}}}{n_c! n_d! n_e!} \times$$

$$\text{Res}_{\mathbf{z}, \mathbf{w}} \left(\prod_{i=1}^{n_c} C^{(-\beta, \alpha)}(z_{n_{ci}}, w_{n_{ci}}) \prod_{i=1}^{n_d} D^{(-\gamma, \alpha)}(z_{n_{di}}, w_{n_{di}}) \prod_{i=1}^{n_e} E^{(-\gamma, \beta)}(z_{n_{ei}}, w_{n_{ei}}) p_{n_c, n_d, n_e} \right) \text{ and}$$

$$p_{n_c, n_d, n_e} = \prod_{1 \leq i < j \leq n_c} (w_{c_i} - w_{c_j})(z_{c_i} - z_{c_j}) \prod_{1 \leq i < j \leq n_d} (w_{d_i} - w_{d_j})(z_{d_i} - z_{d_j}) \times \\ \prod_{1 \leq i < j \leq n_e} (w_{e_i} - w_{e_j})(z_{e_i} - z_{e_j}) \prod_{i=1}^{n_c} \prod_{j=1}^{n_d} (w_{c_i} - w_{d_j}) \prod_{i=1}^{n_d} \prod_{j=1}^{n_e} (z_{d_i} - z_{e_j}) \times \frac{1}{\prod_{i=1}^{n_c} \prod_{j=1}^{n_e} (z_{c_i} - w_{e_j})}.$$

Here, $\text{Res}_{\mathbf{z}, \mathbf{w}}$ means that we successively take the residue over each of the z_i and w_i variables. That is,

$$\text{Res}_{\mathbf{z}, \mathbf{w}} = \text{Res}_{z_{c_1}} \cdots \text{Res}_{z_{c_{n_c}}} \text{Res}_{z_{d_1}} \cdots \text{Res}_{z_{d_{n_d}}} \text{Res}_{z_{e_1}} \cdots \text{Res}_{z_{e_{n_e}}}.$$

$$C^{(-\beta, \alpha)}(z, w) = \sum_{i, j \in \mathbb{Z}} c_{i-\beta, j+\alpha} z^{-i-1} w^{-j-1} \text{ and } D^{(-\gamma, \alpha)} \text{ and } E^{(-\gamma, \beta)} \text{ are similarly defined.}$$

Proof. The proof of the above is analogous to the proof of the formula for the tau-functions in the \widehat{GL}_3 case, which can be found in [AB]. The main difference to clarify here, is the effect that conjugating $g_{C,D,E}$ by the Q_i s has in terms of shifts. We note that

$$(40) \quad Q_0 g Q_0^{-1} = I + \sum_{i, j \in \mathbb{Z}} c_{i, j} E_{10}^{-i-1, j-1} + \sum_{i, j \in \mathbb{Z}} d_{i, j} E_{20}^{-i-1, j-1} + \sum_{i, j \in \mathbb{Z}} e_{i, j} E_{21}^{-i-1, j} = \\ I + \sum_{i, j \in \mathbb{Z}} c_{i, j+1} E_{10}^{-i-1, j} + \sum_{i, j \in \mathbb{Z}} d_{i, j+1} E_{20}^{-i-1, j} + \sum_{i, j \in \mathbb{Z}} e_{i, j} E_{21}^{-i-1, j}.$$

$$(41) \quad Q_1 g Q_1^{-1} = I + \sum_{i, j \in \mathbb{Z}} c_{i, j} E_{10}^{-i-2, j} + \sum_{i, j \in \mathbb{Z}} d_{i, j} E_{20}^{-i-1, j} + \sum_{i, j \in \mathbb{Z}} e_{i, j} E_{21}^{-i-1, j-1} = \\ I + \sum_{i, j \in \mathbb{Z}} c_{i-1, j} E_{10}^{-i-1, j} + \sum_{i, j \in \mathbb{Z}} d_{i, j} E_{20}^{-i-1, j} + \sum_{i, j \in \mathbb{Z}} e_{i, j+1} E_{21}^{-i-1, j}.$$

$$(42) \quad Q_2 g Q_2^{-1} = I + \sum_{i, j \in \mathbb{Z}} c_{i, j} E_{10}^{-i-1, j} + \sum_{i, j \in \mathbb{Z}} d_{i, j} E_{20}^{-i-2, j} + \sum_{i, j \in \mathbb{Z}} e_{i, j} E_{21}^{-i-2, j} = \\ I + \sum_{i, j \in \mathbb{Z}} c_{i, j} E_{10}^{-i-1, j} + \sum_{i, j \in \mathbb{Z}} d_{i-1, j} E_{20}^{-i-1, j} + \sum_{i, j \in \mathbb{Z}} e_{i-1, j} E_{21}^{-i-1, j}.$$

□

From Lemma 6.1 we have

$$(43) \quad \langle v_0, : \psi_b^+(w) \psi_a^-(z) : T_2^{-\ell} T_1^{-k} g v_0 \rangle / \tau_{k, \ell}.$$

Using (43), we obtain the following formulas for $g_{ab}(z, w)$, the analogue of the formulas given above for the $GL_\infty^{(2)}$ case. We note that, as in the $GL_\infty^{(2)}$ case, knowing the formulas for the $g_{ii}(z, w)$ is not necessary. Therefore, the normal ordering in the formula can be ignored, since here we only list the $g_{ab}(z, w)$ terms. We have the following:

Theorem 11.7.

$$(44) \quad g_{01}(z, w) = \frac{S_c^{+1,0}(w)S_c^{0,+1}(z)S_d^{0,+1}(z)S_e^{0,-1}(w)\tau_{k-1,\ell}}{zw\tau_{k,\ell}}$$

$$(45) \quad g_{10}(z, w) = \frac{S_c^{-1,0}(z)S_c^{0,-1}(w)S_d^{0,-1}(w)S_e^{0,+1}(z)\tau_{k+1,\ell}}{zw\tau_{k,\ell}}$$

$$(46) \quad g_{12}(z, w) = (-1)^k \frac{S_c^{-1,0}(z)S_d^{+1,0}(w)S_e^{+1,0}(w)S_e^{0,+1}(z)\tau_{k,\ell-1}}{zw\tau_{k,\ell}}$$

$$(47) \quad g_{21}(z, w) = (-1)^k \frac{S_c^{+1,0}(w)S_d^{-1,0}(z)S_e^{-1,0}(z)S_e^{0,-1}(w)\tau_{k,\ell+1}}{zw\tau_{k,\ell}}$$

$$(48) \quad g_{02}(z, w) = (-1)^k \frac{S_c^{0,+1}(z)S_d^{+1,0}(w)S_d^{0,+1}(z)S_e^{+1,0}(w)\tau_{k-1,\ell-1}}{zw\tau_{k,\ell}}$$

$$(49) \quad g_{20}(z, w) = (-1)^{k+1} \frac{S_c^{0,-1}(w)S_d^{-1,0}(z)S_d^{0,-1}(w)S_e^{-1,0}(z)\tau_{k+1,\ell+1}}{zw\tau_{k,\ell}}$$

Here $S_x^{\pm,0}(z)$ and $S_x^{0,\pm}(z)$ act trivially on any y_i where $y \neq x$. When $y = x$, they act analogously to the shift fields defined by (19) for the $\widehat{GL}_\infty^{(2)}$ case.

The above nonnegative elements give us our difference relations. Let's first write down the relations we obtain from the fact that the coefficients of the $E_{ab}^{-1,0}$ where $a \neq b$ in each of the above are 0.

We use similar notation to that defined earlier for the $\widehat{GL}_\infty^{(2)}$ case, denoting $h_{ab}^{i,j,[k,\ell](\alpha,\beta,\gamma)}$ to be the coefficient of $E_{ab}^{-i-1,j}$ in $g_-^{[k,\ell](\alpha,\beta,\gamma)}$, and let $h_{ab}^{[k,\ell](\alpha,\beta,\gamma)} = h_{ab}^{[k,\ell](\alpha,\beta,\gamma)}$

(1) By Lemma 8.1,

$$(g_-^{[k,\ell](\alpha,\beta,\gamma)})^{-1}Q_0^{-1}g_-^{[k,\ell](\alpha+1,\beta,\gamma)}$$

is nonnegative, so the coefficient of $E_{12}^{-1,0}$ in this expression is 0. So we have

$$-h_{12}^{[k,\ell](\alpha,\beta,\gamma)} + h_{12}^{[k,\ell](\alpha+1,\beta,\gamma)} - h_{10}^{[k,\ell](\alpha,\beta,\gamma)}h_{02}^{[k,\ell](\alpha+1,\beta,\gamma)} = 0.$$

But the $h_{ab}^{[k,\ell](\alpha,\beta,\gamma)}$ are the coefficients of $z^{-1}w^{-1}$ in the $g_{ab}(z, w)$ for $g_-^{[k,\ell](\alpha,\beta,\gamma)}$. So this gives

$$-(-1)^k \frac{\tau_{k,\ell-1}^{(\alpha,\beta,\gamma)}}{\tau_{k,\ell}^{(\alpha,\beta,\gamma)}} + (-1)^k \frac{\tau_{k,\ell-1}^{(\alpha+1,\beta,\gamma)}}{\tau_{k,\ell}^{(\alpha+1,\beta,\gamma)}} - \frac{\tau_{k+1,\ell-1}^{(\alpha,\beta,\gamma)}}{\tau_{k,\ell}^{(\alpha,\beta,\gamma)}} (-1)^k \frac{\tau_{k-1,\ell-1}^{(\alpha+1,\beta,\gamma)}}{\tau_{k,\ell}^{(\alpha+1,\beta,\gamma)}} = 0.$$

Bringing all terms under the same denominator, we have

$$(50) \quad -\tau_{k,\ell-1}^{(\alpha,\beta,\gamma)}\tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} + \tau_{k,\ell-1}^{(\alpha+1,\beta,\gamma)}\tau_{k,\ell}^{(\alpha,\beta,\gamma)} - \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)}\tau_{k-1,\ell-1}^{(\alpha+1,\beta,\gamma)} = 0$$

(2) By (8.1),

$$(g_-^{[k,\ell](\alpha,\beta,\gamma)})^{-1}Q_0^{-1}g_-^{[k+1,\ell](\alpha,\beta+1,\gamma)}$$

is nonnegative. Using the fact that the coefficient of $E_{12}^{-1,0}$ in this expression is 0, we obtain

$$-h_{12}^{[k,\ell](\alpha,\beta,\gamma)} + h_{12}^{[k+1,\ell](\alpha,\beta+1,\gamma)} - h_{10}^{[k,\ell](\alpha,\beta,\gamma)}h_{02}^{[k+1,\ell](\alpha,\beta+1,\gamma)} = 0,$$

which gives

$$-(-1)^k \frac{\tau_{k,\ell-1}^{(\alpha,\beta,\gamma)}}{\tau_{k,\ell}^{(\alpha,\beta,\gamma)}} + (-1)^{k+1} \frac{\tau_{k+1,\ell-1}^{(\alpha,\beta+1,\gamma)}}{\tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma)}} - \frac{\tau_{k+1,\ell}^{(\alpha,\beta,\gamma)}}{\tau_{k,\ell}^{(\alpha,\beta,\gamma)}} (-1)^{k+1} \frac{\tau_{k,\ell-1}^{(\alpha,\beta+1,\gamma)}}{\tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma)}} = 0.$$

Bringing all terms under the same denominator, this gives

$$(51) \quad \tau_{k+1,\ell-1}^{(\alpha,\beta+1,\gamma)} \tau_{k,\ell}^{(\alpha,\beta,\gamma)} + \tau_{k,\ell-1}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma)} = \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell-1}^{(\alpha,\beta+1,\gamma)}.$$

(3) By (8.1),

$$(g_-^{[k,\ell](\alpha,\beta,\gamma)})^{-1} Q_1^{-1} g_-^{[k,\ell+1](\alpha,\beta,\gamma+1)}$$

is nonnegative. So the coefficient of $E_{02}^{-1,0}$ in this expression is 0, which gives

$$-h_{02}^{[k,\ell](\alpha,\beta,\gamma)} + h_{02}^{[k,\ell+1](\alpha,\beta,\gamma+1)} - h_{01}^{[k,\ell](\alpha,\beta,\gamma)} h_{12}^{[k,\ell+1](\alpha,\beta,\gamma+1)} = 0,$$

which is

$$-(-1)^k \frac{\tau_{k-1,\ell-1}^{(\alpha,\beta,\gamma)}}{\tau_{k,\ell}^{(\alpha,\beta,\gamma)}} + (-1)^k \frac{\tau_{k-1,\ell}^{(\alpha,\beta,\gamma+1)}}{\tau_{k,\ell+1}^{(\alpha,\beta,\gamma+1)}} - \frac{\tau_{k-1,\ell}^{(\alpha,\beta,\gamma)}}{\tau_{k,\ell}^{(\alpha,\beta,\gamma)}} (-1)^k \frac{\tau_{k,\ell}^{(\alpha,\beta,\gamma+1)}}{\tau_{k,\ell+1}^{(\alpha,\beta,\gamma+1)}} = 0.$$

This is then equivalent to

$$(52) \quad \tau_{k-1,\ell-1}^{(\alpha,\beta,\gamma)} \tau_{k,\ell+1}^{(\alpha,\beta,\gamma+1)} - \tau_{k-1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell}^{(\alpha,\beta,\gamma)} + \tau_{k-1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha,\beta,\gamma+1)} = 0.$$

Next, we prove that the following theorem:

Theorem 11.8. *The tau-functions satisfy*

$$(53) \quad \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} = \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k,\ell}^{(\alpha,\beta+1,\gamma+1)} + \tau_{k+1,\ell+1}^{(\alpha,\beta+1,\gamma+1)} \tau_{k-1,\ell-1}^{(\alpha+1,\beta,\gamma)} - \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k-1,\ell}^{(\alpha+1,\beta,\gamma)}.$$

Proof. The proof of this theorem uses formulas for differences between $h_{00}^{[k,\ell](\alpha,\beta,\gamma)}$ s. We comment that we have analogous formulas for the $h_{11}^{[k,\ell](\alpha,\beta,\gamma)}$ s and $h_{22}^{[k,\ell](\alpha,\beta,\gamma)}$ s, but since we don't need them for this theorem, we omit them. Using the nonnegativity of

$$(g_-^{[k,\ell](\alpha,\beta,\gamma)})^{-1} Q_1^{-1} g_-^{[k,\ell](\alpha,\beta+1,\gamma)},$$

we obtain

$$(54) \quad h_{00}^{[k,\ell](\alpha,\beta+1,\gamma)} - h_{00}^{[k,\ell](\alpha,\beta,\gamma)} = \frac{\tau_{k-1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma)}}{\tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha,\beta+1,\gamma)}}.$$

The nonnegativity of $(g_-^{[k+1,\ell](\alpha,\beta,\gamma)})^{-1} Q_1^{-1} g_-^{[k,\ell](\alpha+1,\beta,\gamma)}$ implies

$$(55) \quad h_{00}^{[k,\ell](\alpha+1,\beta,\gamma)} - h_{00}^{[k+1,\ell](\alpha,\beta,\gamma)} = \frac{\tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta,\gamma)}}{\tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)}}.$$

Finally, the nonnegativity of $(g_-^{[k,\ell](\alpha,\beta,\gamma)})^{-1} Q_2^{-1} g_-^{[k,\ell](\alpha,\beta,\gamma+1)}$ gives

$$(56) \quad h_{00}^{[k,\ell](\alpha,\beta,\gamma)} - h_{00}^{[k,\ell](\alpha,\beta,\gamma+1)} = \frac{\tau_{k-1,\ell-1}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell+1}^{(\alpha,\beta,\gamma+1)}}{\tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma+1)}}.$$

We have the following identity, which holds trivially:

$$(57) \quad - (h_{00}^{[k+1,\ell](\alpha,\beta,\gamma)} - h_{00}^{[k+1,\ell](\alpha,\beta,\gamma+1)}) + (h_{00}^{[k,\ell](\alpha+1,\beta+1,\gamma)} - h_{00}^{[k,\ell](\alpha+1,\beta+1,\gamma+1)}) \\ - (h_{00}^{[k,\ell](\alpha+1,\beta+1,\gamma)} - h_{00}^{[k,\ell](\alpha+1,\beta,\gamma)}) + (h_{00}^{[k,\ell](\alpha+1,\beta+1,\gamma+1)} - h_{00}^{[k+1,\ell](\alpha,\beta+1,\gamma+1)}) \\ + (h_{00}^{[k+1,\ell](\alpha,\beta+1,\gamma+1)} - h_{00}^{[k+1,\ell](\alpha,\beta,\gamma+1)}) - (h_{00}^{[k,\ell](\alpha+1,\beta,\gamma)} - h_{00}^{[k+1,\ell](\alpha,\beta,\gamma)}) = 0$$

Using (54), (55), and (56), we see that this is equivalent to

$$(58) \quad - \frac{\tau_{k,\ell-1}^{(\alpha,\beta,\gamma)} \tau_{k+2,\ell+1}^{(\alpha,\beta,\gamma+1)}}{\tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)}} + \frac{\tau_{k-1,\ell-1}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell+1}^{(\alpha+1,\beta+1,\gamma+1)}}{\tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)}} - \frac{\tau_{k-1,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma)}}{\tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)}} \\ + \frac{\tau_{k,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)}}{\tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)}} + \frac{\tau_{k,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+2,\ell}^{(\alpha,\beta+1,\gamma+1)}}{\tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)}} - \frac{\tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta,\gamma)}}{\tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)}} = 0$$

Bringing all terms under the same denominator, the vanishing of the numerator gives:

$$(59) \quad - \tau_{k,\ell-1}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k+2,\ell+1}^{(\alpha,\beta,\gamma+1)} \\ + \tau_{k-1,\ell-1}^{(\alpha+1,\beta+1,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k+1,\ell+1}^{(\alpha+1,\beta+1,\gamma+1)} \\ - \tau_{k-1,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma)} \\ + \tau_{k,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \\ + \tau_{k,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+2,\ell}^{(\alpha,\beta+1,\gamma+1)} \\ - \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha+1,\beta,\gamma)} = 0.$$

To prove our theorem, we will first need to make several substitutions into

$$(60) \quad \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} (\tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k,\ell}^{(\alpha,\beta+1,\gamma+1)} + \\ \tau_{k+1,\ell+1}^{(\alpha,\beta+1,\gamma+1)} \tau_{k-1,\ell-1}^{(\alpha+1,\beta,\gamma)} - \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k-1,\ell}^{(\alpha+1,\beta,\gamma)}) = \\ \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} (\tau_{k+1,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} + \\ \tau_{k+2,\ell+1}^{(\alpha,\beta+1,\gamma+1)} \tau_{k,\ell-1}^{(\alpha+1,\beta,\gamma)} - \tau_{k+2,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)})$$

to show that it is equivalent to (59). We note that if (53) holds for $k = 0$, (60) implies that it holds for $k = 1$. (53) holds for $k = 0$, since when $k = 0$, it reduces to

$$\tau_{0,\ell}^{(\alpha,\beta,\gamma)} \tau_{0,\ell}^{(\alpha+1,\beta+1,\gamma+1)} = \tau_{0,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{0,\ell}^{(\alpha,\beta+1,\gamma+1)},$$

which holds since

$$\tau_{0,\ell}^{(\alpha,\beta,\gamma)} = c_{0,0,n_e}^{(\alpha,\beta,\gamma)} = \text{Res}_{\mathbf{z}, \mathbf{w}} \left(\prod_{i=1}^{n_e} E^{(-\gamma, \beta)}(z_{n_e i}, w_{n_e i}) \prod_{1 \leq i < j \leq n_e} (w_{e_i} - w_{e_j})(z_{e_i} - z_{e_j}) \right),$$

does not depend on the α parameter. Therefore, since we know that (59) is true, we need only show that (60) is equivalent to (59) and our theorem will be proven.

(60) is

$$\begin{aligned}
& \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k,\ell}^{(\alpha,\beta+1,\gamma+1)} + \\
& \quad + \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell+1}^{(\alpha,\beta+1,\gamma+1)} \tau_{k-1,\ell-1}^{(\alpha+1,\beta,\gamma)} \\
& - \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k-1,\ell}^{(\alpha+1,\beta,\gamma)} - \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \\
& \quad - \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+2,\ell+1}^{(\alpha,\beta+1,\gamma+1)} \tau_{k,\ell-1}^{(\alpha+1,\beta,\gamma)} \\
& \quad + \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+2,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} = 0.
\end{aligned}$$

As we proceed in the following, we will hilight in **red** the terms that agree with (59). The terms used in substitutions will be hilighted in **blue**.

Begin by multiplying both sides of the equation by $\tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)}$:

$$\begin{aligned}
& \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k,\ell}^{(\alpha,\beta+1,\gamma+1)} + \\
& \quad + \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell+1}^{(\alpha,\beta+1,\gamma+1)} \tau_{k-1,\ell-1}^{(\alpha+1,\beta,\gamma)} \\
& - \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k-1,\ell}^{(\alpha+1,\beta,\gamma)} \\
& - \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \\
& - \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+2,\ell+1}^{(\alpha,\beta+1,\gamma+1)} \tau_{k,\ell-1}^{(\alpha+1,\beta,\gamma)} \\
& \quad + \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+2,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} = 0.
\end{aligned}$$

By (52), $\tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell}^{(\alpha,\beta,\gamma)} = \tau_{k,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} - \tau_{k+1,\ell+1}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell-1}^{(\alpha,\beta,\gamma)}$. This gives us

$$\begin{aligned}
& \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k,\ell}^{(\alpha,\beta+1,\gamma+1)} + \\
& \quad + \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell+1}^{(\alpha,\beta+1,\gamma+1)} \tau_{k-1,\ell-1}^{(\alpha+1,\beta,\gamma)} \\
& - \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k-1,\ell}^{(\alpha+1,\beta,\gamma)} \\
& - \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha+1,\beta,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma+1)} \\
& - \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+2,\ell+1}^{(\alpha,\beta+1,\gamma+1)} \tau_{k,\ell-1}^{(\alpha+1,\beta,\gamma)} \\
& \quad + \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+2,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} \\
& \quad - \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell+1}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell-1}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+2,\ell}^{(\alpha,\beta+1,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma)} = 0.
\end{aligned}$$

Applying (52) again, we substitute

$$\tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma+1)} = \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k+1,\ell}^{(\alpha+1,\beta+1,\gamma)} - \tau_{k+1,\ell+1}^{(\alpha+1,\beta+1,\gamma+1)} \tau_{k,\ell-1}^{(\alpha+1,\beta+1,\gamma)}.$$

Theorem 11.9. *The tau-functions satisfy*

$$(61) \quad \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} = \tau_{k,\ell}^{(\alpha,\beta+1,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma+1)} - \tau_{k,\ell+1}^{(\alpha+1,\beta,\gamma+1)} \tau_{k,\ell-1}^{(\alpha,\beta+1,\gamma)} - \tau_{k-1,\ell}^{(\alpha+1,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma)},$$

Proof. We prove this in the same way we proved Theorem 53, this time by expressing the trivially satisfied relation,

$$(62) \quad (h_{11}^{[k,\ell+1](\alpha,\beta,\gamma+1)} - h_{11}^{[k,\ell+1](\alpha,\beta,\gamma)}) + (h_{11}^{[k,\ell](\alpha+1,\beta+1,\gamma)} - h_{11}^{[k,\ell](\alpha+1,\beta+1,\gamma+1)}) - (h_{11}^{[k,\ell](\alpha+1,\beta+1,\gamma)} - h_{11}^{[k,\ell](\alpha,\beta+1,\gamma)}) + (h_{11}^{[k,\ell](\alpha+1,\beta+1,\gamma+1)} - h_{11}^{[k,\ell+1](\alpha+1,\beta,\gamma+1)}) + (h_{11}^{[k,\ell+1](\alpha+1,\beta,\gamma+1)} - h_{11}^{[k,\ell+1](\alpha,\beta,\gamma+1)}) - (h_{11}^{[k,\ell](\alpha,\beta+1,\gamma)} - h_{11}^{[k,\ell+1](\alpha,\beta,\gamma)}) = 0,$$

in terms of tau-functions. As in the proof of Theorem 11.8, we make several substitutions, ultimately showing that (62) is equivalent to

$$(63) \quad \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} (\tau_{k,\ell+1}^{(\alpha,\beta+1,\gamma)} \tau_{k,\ell+1}^{(\alpha+1,\beta,\gamma+1)} - \tau_{k,\ell+2}^{(\alpha+1,\beta,\gamma+1)} \tau_{k,\ell}^{(\alpha,\beta+1,\gamma)} - \tau_{k-1,\ell+1}^{(\alpha+1,\beta,\gamma+1)} \tau_{k+1,\ell+1}^{(\alpha,\beta+1,\gamma)}) = \tau_{k,\ell+1}^{(\alpha,\beta,\gamma)} \tau_{k,\ell+1}^{(\alpha+1,\beta+1,\gamma+1)} (\tau_{k,\ell}^{(\alpha,\beta+1,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta,\gamma+1)} - \tau_{k,\ell+1}^{(\alpha+1,\beta,\gamma+1)} \tau_{k,\ell-1}^{(\alpha,\beta+1,\gamma)} - \tau_{k-1,\ell}^{(\alpha+1,\beta,\gamma+1)} \tau_{k+1,\ell}^{(\alpha,\beta+1,\gamma)})$$

and using induction, noting that (61) holds for $\ell = 0$ and that (63) implies that if (61) holds for some ℓ , it holds for $\ell + 1$. \square

Theorem 11.10. *The tau-functions satisfy*

$$(64) \quad \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} = \tau_{k,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} + \tau_{k-1,\ell-1}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell+1}^{(\alpha,\beta,\gamma+1)} - \tau_{k,\ell-1}^{(\alpha+1,\beta+1,\gamma)} \tau_{k,\ell+1}^{(\alpha,\beta,\gamma+1)}.$$

Proof. We prove this in the same way we proved the previous two theorems, this time by expressing the trivially satisfied relation,

$$(65) \quad - (h_{22}^{[k,\ell+1](\alpha,\beta,\gamma+1)} - h_{22}^{[k,\ell](\alpha,\beta,\gamma)}) + (h_{22}^{[k,\ell+1](\alpha+1,\beta+1,\gamma+1)} - h_{22}^{[k,\ell](\alpha+1,\beta+1,\gamma)}) + (h_{22}^{[k,\ell](\alpha+1,\beta+1,\gamma)} - h_{22}^{[k,\ell](\alpha+1,\beta,\gamma)}) + (h_{22}^{[k,\ell](\alpha+1,\beta,\gamma)} - h_{22}^{[k,\ell](\alpha,\beta,\gamma)}) - (h_{22}^{[k,\ell+1](\alpha+1,\beta+1,\gamma+1)} - h_{22}^{[k,\ell+1](\alpha,\beta+1,\gamma+1)}) - (h_{22}^{[k,\ell+1](\alpha,\beta+1,\gamma+1)} - h_{22}^{[k,\ell+1](\alpha,\beta,\gamma+1)}) = 0,$$

in terms of tau-functions. As in the proofs of Theorems 11.8 and 11.9, we make several substitutions, ultimately showing that (65) is equivalent to

$$(66) \quad \tau_{k,\ell}^{(\alpha,\beta,\gamma)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma+1)} (\tau_{k,\ell+1}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell+1}^{(\alpha+1,\beta+1,\gamma)} + \tau_{k-1,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell+2}^{(\alpha,\beta,\gamma+1)} - \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} \tau_{k,\ell+2}^{(\alpha,\beta,\gamma+1)}) = \tau_{k,\ell+1}^{(\alpha,\beta,\gamma)} \tau_{k,\ell+1}^{(\alpha+1,\beta+1,\gamma+1)} (\tau_{k,\ell}^{(\alpha,\beta,\gamma+1)} \tau_{k,\ell}^{(\alpha+1,\beta+1,\gamma)} + \tau_{k-1,\ell-1}^{(\alpha+1,\beta+1,\gamma)} \tau_{k+1,\ell+1}^{(\alpha,\beta,\gamma+1)} - \tau_{k,\ell-1}^{(\alpha+1,\beta+1,\gamma)} \tau_{k,\ell+1}^{(\alpha,\beta,\gamma+1)})$$

and using induction. \square

11.2. **Proof of \widehat{GL}_3 Difference Relations.** In [AB], we proved that

$$V_{[k,\ell]}^{(\alpha+,\beta)} = (g_-^{[k,\ell](\alpha,\beta)})^{-1} Q_0^{-1} g_-^{[k,\ell](\alpha+1,\beta)}$$

and

$$W_{[k-,\ell]}^{(\alpha+,\beta)} = (g_-^{[k,\ell](\alpha,\beta)})^{-1} Q_1^{-1} g_-^{[k-1,\ell](\alpha+1,\beta)}$$

are nonnegative. Let $h_{ab}^{i,[k,\ell](\alpha,\beta)}$ denote the coefficient of $E_{ab}z^{-i-1}$ in $g_-^{[k,\ell](\alpha,\beta)}$. The fact that the coefficient of $E_{02}z^{-1}$ in $W_{[k+1-,\ell]}^{(\alpha+,\beta)}$ is 0 gives us

$$(67) \quad -h_{02}^{[k+1,\ell](\alpha,\beta)} + h_{02}^{[k,\ell](\alpha+1,\beta)} - h_{01}^{[k+1,\ell](\alpha,\beta)} h_{12}^{[k,\ell](\alpha+1,\beta)} = 0$$

$$(68) \quad -(-1)^{k+1} \frac{\tau_{k,\ell-1}^{(\alpha,\beta)}}{\tau_{k+1,\ell}^{(\alpha,\beta)}} + (-1)^k \frac{\tau_{k-1,\ell-1}^{(\alpha+1,\beta)}}{\tau_{k,\ell}^{(\alpha+1,\beta)}} - \frac{\tau_{k,\ell}^{(\alpha,\beta)}}{\tau_{k+1,\ell}^{(\alpha,\beta)}} (-1)^k \frac{\tau_{k,\ell-1}^{(\alpha+1,\beta)}}{\tau_{k,\ell}^{(\alpha+1,\beta)}} = 0.$$

Bringing all terms under the same denominator, we obtain

$$\tau_{k,\ell}^{(\alpha+1,\beta)} \tau_{k,\ell-1}^{(\alpha,\beta)} + \tau_{k+1,\ell}^{(\alpha,\beta)} \tau_{k-1,\ell-1}^{(\alpha+1,\beta)} - \tau_{k,\ell}^{(\alpha,\beta)} \tau_{k,\ell-1}^{(\alpha+1,\beta)} = 0,$$

which is precisely one of the T -system relations we proved in our [AB]. We prove the four-term relation in much the same way we proved the Q -system relation in the \widehat{GL}_2 case, by using the nonnegativity of the V s and W s to find expressions for the $h_{aa}^{[k,\ell](\alpha,\beta)} - h_{aa}^{[k',\ell'](\alpha',\beta')}$ in terms of the tau-functions, where $h_{aa}^{[k,\ell](\alpha,\beta)}$ denotes the coefficient of z^{-1} in $g_-^{[k,\ell](\alpha,\beta)}$.

To derive the four-term relation that depends on shifts in α , we use the fact that

$$h_{00}^{[k,\ell](\alpha,\beta)} + h_{11}^{[k,\ell](\alpha,\beta)} + h_{22}^{[k,\ell](\alpha,\beta)} = 0,$$

to see that

$$\begin{aligned} & - (h_{22}^{[k+1,\ell](\alpha-1,\beta)} - h_{22}^{[k+1,\ell](\alpha,\beta)}) + (h_{22}^{[k,\ell](\alpha,\beta)} - h_{22}^{[k,\ell](\alpha+1,\beta)}) \\ & - (h_{11}^{[k,\ell](\alpha+1,\beta)} - h_{11}^{[k,\ell](\alpha,\beta)}) + (h_{00}^{[k,\ell](\alpha,\beta)} - h_{00}^{[k+1,\ell](\alpha-1,\beta)}) + \\ & (h_{11}^{[k+1,\ell](\alpha,\beta)} - h_{11}^{[k+1,\ell](\alpha-1,\beta)}) - (h_{00}^{[k,\ell](\alpha+1,\beta)} - h_{00}^{[k+1,\ell](\alpha,\beta)}) = 0. \end{aligned}$$

The formulas for the $h_{aa}^{[k,\ell](\alpha,\beta)} - h_{aa}^{[k',\ell'](\alpha',\beta')}$ s in terms of the tau-functions then give

$$(69) \quad -\frac{\tau_{k+2,\ell+1}^{(\alpha-1,\beta)} \tau_{k,\ell-1}^{(\alpha,\beta)}}{\tau_{k+1,\ell}^{(\alpha-1,\beta)} \tau_{k+1,\ell}^{(\alpha,\beta)}} + \frac{\tau_{k+1,\ell+1}^{(\alpha,\beta)} \tau_{k-1,\ell-1}^{(\alpha+1,\beta)}}{\tau_{k,\ell}^{(\alpha+1,\beta)} \tau_{k,\ell}^{(\alpha,\beta)}} - \frac{\tau_{k+1,\ell}^{(\alpha,\beta)} \tau_{k-1,\ell}^{(\alpha+1,\beta)}}{\tau_{k,\ell}^{(\alpha+1,\beta)} \tau_{k,\ell}^{(\alpha,\beta)}} \\ + \frac{\tau_{k+1,\ell}^{(\alpha,\beta)} \tau_{k,\ell}^{(\alpha-1,\beta)}}{\tau_{k,\ell}^{(\alpha,\beta)} \tau_{k+1,\ell}^{(\alpha-1,\beta)}} + \frac{\tau_{k+2,\ell}^{(\alpha-1,\beta)} \tau_{k,\ell}^{(\alpha,\beta)}}{\tau_{k+1,\ell}^{(\alpha-1,\beta)} \tau_{k+1,\ell}^{(\alpha,\beta)}} - \frac{\tau_{k+1,\ell}^{(\alpha+1,\beta)} \tau_{k,\ell}^{(\alpha,\beta)}}{\tau_{k+1,\ell}^{(\alpha,\beta)} \tau_{k,\ell}^{(\alpha+1,\beta)}} = 0,$$

which is precisely the relation we used in our paper to prove that the tau-functions satisfy the α -dependent, four-term quadratic equality.

We similarly prove the β -dependent identities found in [AB].

12. CONJECTURES FOR \widehat{GL}_n TAU-FUNCTION RELATIONS

Let

$h_{ab}^{[\mathbf{k}](\alpha_1, \dots, \alpha_{n-1})}$ be the coefficient of $E_{ab}z^{-1}$ in $g_-^{[\mathbf{k}](\beta_1, \dots, \beta_{n-1})}$,

the negative part of the Birkhoff factorization of

$$T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1} g^{(\beta_1, \dots, \beta_{n-1})},$$

where $\mathbf{k} = k_1, \dots, k_{n-1}$. Then for each i , $1 \leq i \leq n-1$,

$$\begin{aligned} 1 - \frac{(\tau_{\mathbf{k}}^{(\beta_1, \dots, \beta_{n-1})})^2}{\tau_{\mathbf{k}}^{(\beta_1, \dots, \beta_{i-1}, \beta_i-1, \beta_{i+1}, \dots, \beta_{n-1})} \tau_{\mathbf{k}}^{(\beta_1, \dots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \dots, \beta_{n-1})}} &= \\ &= \sum_{j \neq i} h_{ji}^{[\mathbf{k}](\beta_1, \dots, \beta_{i-1}, \beta_i-1, \beta_{i+1}, \dots, \beta_{n-1})} h_{ij}^{[\mathbf{k}](\beta_1, \dots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \dots, \beta_{n-1})}. \end{aligned}$$

13. CONJECTURES FOR THE $\widehat{GL}_\infty^{(n)}$ TAU-FUNCTION RELATIONS

Let

$h_{ab}^{[\mathbf{k}](\beta_0, \dots, \beta_{n-1})}$ be the coefficient of E_{ab}^{-10} in $g_-^{[\mathbf{k}](\beta_0, \dots, \beta_{n-1})}$,

the negative part of the Gauss factorization of

$$T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1} g^{(\beta_0, \dots, \beta_{n-1})},$$

where $\mathbf{k} = k_1, \dots, k_{n-1}$. Then for each i , $0 \leq i \leq n-1$,

$$\begin{aligned} 1 - \frac{\tau_{\mathbf{k}}^{(\beta_0, \dots, \beta_{n-1})} \tau_{\mathbf{k}}^{(\beta_0+1, \dots, \beta_{n-1}+1)}}{\tau_{\mathbf{k}}^{(\beta_0, \dots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \dots, \beta_{n-1})} \tau_{\mathbf{k}}^{(\beta_0+1, \dots, \beta_{i-1}+1, \beta_i, \beta_{i+1}+1, \dots, \beta_{n-1}+1)}} &= \\ &= \sum_{j \neq i} h_{ji}^{[\mathbf{k}](\beta_0+1, \dots, \beta_{i-1}+1, \beta_i, \beta_{i+1}+1, \dots, \beta_{n-1}+1)} h_{ij}^{[\mathbf{k}](\beta_0, \dots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \dots, \beta_{n-1})}. \end{aligned}$$

Remark 13.1. We have already proven the above conjectures for the cases $n = 2, 3$, since the $n = 2$ case gives the $2Q$ -system and the $n = 3$ case gives the four term relations of the $3Q$ -system. Similarly, the $n = 2$ case of the $\widehat{GL}_\infty^{(n)}$ conjecture gives the $2T$ -system and the $n = 3$ case gives the four term $3T$ -system relations.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801, USA
E-mail address: addabbo2@illinois.edu, bergv@illinois.edu